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CHAINS AND UNIONS OF PRIME SUBMODULES

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ABSTRACT

Abstract. Let R be a commutative ring with identity and let M be a unital R-module. In this paper we study the various properties of prime submodules. Also we give a new equivalent conditions for a minimal prime submodules of an R-module to be a finite set, whenever R is a Noetherian ring. Finally we prove the Prime avoidance Theorem for modules in different states.

Keyword:
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1. Introduction

Throughout this paper, let R be a commutative ring (with identity) and M be a unital R-module. A proper submodules N of M with $N :_R M = p$ is said to be prime or p-prime (p a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ *or* $r \in p$. Another equivalent notion of prime submodules was first introduced and systematically studied in [**5**]. Prime submodules have been studied by several authors; see, for example, [**3**], [**1**], [**6**], [**8**], [**9**], [**10**], [**11**] and [**13**]. In section 2 we study the chain of prime submodules and we shall improve the results given in [**10**]. The Prime avoidance Theorem states that if an ideal I of a ring is contained in the union of finite number of prime ideals, then I must be contained in one of them. This result's generalization for the non-commutative case has been proved in [7]. In section 2, we generalize this theorem for modules in different states. In section 4 we prove some new results about the finiteness of the set of minimal prime submodules of an R-module. Also we introduce the concept of arithmetic rank of a submodule of a Noetherian module and we give an upper bound for it. Throughout, for any ideal b of R, the radical of b, denoted by *Rad(b)*, is defined to be the set $\{x \in R : x^n \in b$ *for some* $n \in N\}$ and we denote $\{p \in Spec(R) : p \supseteq b\}$ *by V(b), where Spec(R)* denotes the set of all prime ideals of R. The symbol $\subseteq$ denotes containment and $\subset$ denotes proper containment for sets. If N is a submodule of M, we write $N \leq M$. We denote the annihilator of a factor module M/N of M by $(N :_R M)$. The set of all maximal ideals of R is denoted by Max(R). For any ideal I of a ring R and for any R-module M, $\Gamma_I(M)$ is defined to be the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. For any unexplained notation and terminology we refer the reader to [**4**], [**12**] and [**15**].

2. Chains of prime submodules

The results of this section are generalizations of the some results given in [10] and [3]. First we need the following definition.

Definition 2.1. Let R be a Noetherian ring and M be a finitely generated R-module. For each $p \in$ Spec(R) we define $\lambda_p(M)$ as following:

$$\lambda_p(M) = \dim_{R_p/pR_p}(M_p/pM_p).$$

Remark 2.2. Let R be a Noetherian ring and M be a finitely generated R-module. For each $p \in \mathrm{Spec}(R)$, $\lambda_p(M)$ is the number of elements of any minimal generator set of the Rp-module Mp and so $\lambda_p(M) < \infty$. Also we have $\lambda_p(M) = 0$ if and only if $p \notin \mathrm{Supp}(M)$. Moreover, for any pair $q \subseteq p$ of prime ideals of R it is easy to see that $\lambda_q(M) \leq \lambda_p(M)$.

The following description of prime submodules will be useful in this paper.

Lemma 2.3. Let R be a Noetherian ring and $p \in \mathrm{Spec}(R)$. Let M be a finitely generated R-module and N be a proper submodule of M. Then the followings are equivalent:

(i) N is p-prime submodule of M.

(ii) $\mathrm{Ass}_R(M/N) = \{p\}$ *and* $(N :_R M) = p$.

(iii) $(N :_R x) = p$, for each $x \in M/N$.

Proof. Easily follows from definition.

The following theorem is the first main result of this paper and a generalization of [10, Lemma 2.6].

Theorem 2.4. Let R be a Noetherian ring and $p \in$ Supp(M). Let M be a finitely generated R-module. Then the following statements hold:

(i) The length of any chain of p-prime submodules of M is bounded from above by $\lambda_p(M) - 1$.

(ii) There is a chain of p-prime submodules of M, which is of length $\lambda_p(M) - 1$.

(iii) Any saturated maximal chain of p-prime submodules of M is of length $\lambda_p(M) - 1$.

Proof. (i) Let $n := \lambda_p(M)$. Then it follows from the hypothesis $p \in$ Supp(M) that $n > 0$. Suppose the contrary be true. Then there exist a chain of p-prime submodules of M as;

$N_0 \subset N_1 \subset \cdots \subset N_n$

By Lemma 2.3 we have $p \in \mathrm{Supp}(M/N_n)$ and so $l_{R_p}((M/N_n)_p) \geq 1$. On the other hand since by assumption we have $(N_0 :_R M) = p$, it follows that there is an exact sequence

$M/pM \rightarrow M/N_0 \rightarrow 0.$

Hence we have the following exact sequence:

$(M/pM)_p \rightarrow (M/N_0)_p \rightarrow 0.$

Therefore, it follows from definition that

$l_{R_p}((M/N_0)_p) = dim_{R_p/pR_p}((M/N_0)_p) \leq \lambda_p(M) = n.$

On *the other hand for each* $0 \leq i \leq n-1$ *there is an exact sequence*

$0 \rightarrow N_{i+1}/N_i \rightarrow M/N_i.$

But, since $N_{i+1}/N_i \neq 0$, it follows from Lemma 2.3 and above exact sequence that

$\emptyset \neq \mathrm{Ass}_R(N_{i+1}/N_i) \subseteq \mathrm{Ass}_R(M/N_i) = \{p\},$

Which implies that $\mathrm{Ass}_R(N_{i+1}/N_i) = \{p\}$. In particular $p \in \mathrm{Supp}(N_{i+1}/N_i)$, and so $(N_{i+1}/N_i)_p \neq 0$.Consequently $l_{R_p}((N_{i+1}/N_i)_p) \geq 1$. Whence, we have

$n = \sum_{i=0}^{n-1} 1 \leq \sum_{i=0}^{n-1} l_{R_p}((N_{i+1}/N_i)_p) = l_{R_p}((N_n/N_0)_p) \leq l_{R_p}((M/N_0)_p) - 1 \leq n-1$

,Which is a contradiction.

(ii) Let $\lambda_p(M) = n$. Then $n > 0$. As $p \in$ Supp(M) it follows that $(pM :_R M) = p$. Therefore, $p \in \mathrm{Ass}_R(M/pM)$. Let $N_0 = pM$, whenever $\mathrm{Ass}_R(M/pM) = \{p\}$. In other case suppose

$\mathrm{Ass}_R(M/pM)\backslash\{p\} := \{q_1, \cdots, q_k\}.$

Let $I = \bigcap_{j=1}^{k} q_j$ and $N_0/pM := \Gamma_I(M/pM)$. Then we have

$\mathrm{Ass}_R(M/N_0) = \mathrm{Ass}_R((M/pM)/\Gamma_I(M/pM)) = \mathrm{Ass}_R(M/pM)\backslash V(I)$

.But, since for each $1 \leq j \leq k$ we have $\mathrm{Ann}_R(M/pM) = p \subseteq q_j$ and $q_j \neq p$, it follows that $p \notin V(q_j)$. Therefore

$p \notin \bigcup_{j=1}^{k} V(q_j) = V(\bigcap_{j=1}^{k} q_j) = V(I).$

Therefore

$\mathrm{Ass}_R(M/N_0) = \mathrm{Ass}_R(M/pM)\backslash V(I) = \{p\},$

Which results $\mathrm{Ann}_R(M/N_0) \subseteq p$. Therefore, we have $p = (pM :_R M) \subseteq (N_0 :_R M) \subseteq p$ and so $(N_0 :_R M) = p$. Also as

$\mathrm{Ass}_R(N_0/pM) = \mathrm{Ass}_R(\Gamma_I(M/pM)) = \mathrm{Ass}_R(M/pM) \cap V(I)$

,

It follows that $p \notin \mathrm{Supp}(N_0/pM)$ and hence $(N_0/pM)_p = 0$. Now in both cases it follows from Lemma 2.3 that N_0 is a p-prime submodule of M. We shall construct the chain $N_0 \subset \cdots \subset N_{n-1}$ of p-prime submodules of M such that $l_{R_p}((N_{i+1}/N_i)_p) = 1$, for each $0 \le i \le n-2$, by an inductive process. To do this end, assume $0 \le j < n-1$, and that we have already constructed $N_0 \subset N_1 \subset \cdots \subset N_j$. We show how to construct N_{j+1}. To do this, since by definition $M \neq N_j$ it follows that there is an element $x \in M \backslash N_j$. Let $L := Rx + N_j$. In view of Lemma 2.3 we have $L/N_j \cong R/p$. In particular we have $l_{R_p}((L/N_j)_p) = 1$. By inductive hypothesis we have

$$l_{R_p}((M/L)_p) = l_{R_p}((M/N_0)_p) - l_{R_p}((L/N_0)_p) = l_{R_p}((M/pM)_p) - [\, l_{R_p}((L/N_j)_p) + \textstyle\sum_{i=1}^{j-1} l_{R_p}((N_{i+1}/N_i)_p)] =$$

$n - (1 + j) = n - j - 1 > 0$.

Therefore, $(M/L)_p \neq 0$. Now it is easy to see that $(L :_R M) = p$, and so $p \in \mathrm{Ass}_R(M/L)$. Let $N_{j+1} = L$ whenever $\mathrm{Ass}_R(M/L) = \{p\}$. In other case suppose

$\mathrm{Ass}_R(M/L)\backslash\{p\} := \{q_1{}', \cdots, q_t{}'\}$.

Let $J = \bigcap_{i=1}^{t} q'_i$ and $N_{j+1}/L := \Gamma_J(M/L)$. Then we have

$\mathrm{Ass}_R(M/N_{j+1}) = \mathrm{Ass}_R((M/L)/\Gamma_J(M/L)) = \mathrm{Ass}_R(M/L)\backslash V(J)$

.

But, since for each $1 \le i \le t$ we have $\mathrm{Ann}_R(M/L) = p \subseteq q_i{}'$ and $p \neq q_i{}'$, it follows that $p \notin V(q_i')$. Therefore,

$\mathrm{Ass}_R(M/N_{j+1}) = \mathrm{Ass}_R(M/L)\backslash V(J) = \{p\}$,

Which results $\mathrm{Ann}_R(M/N_{j+1}) \subseteq p$. Therefore, we have $p = (L :_R M) \subseteq (N_{j+1} :_R M) \subseteq p$ and so $(N_{j+1} :_R M) = p$. Also as

$\mathrm{Ass}_R(N_{j+1}/L) = \mathrm{Ass}_R(\Gamma_J(M/L)) = \mathrm{Ass}_R(M/L) \cap V(J)$,

it follows that $p \notin \mathrm{Supp}(N_{j+1}/L)$ and hence $(N_{j+1}/L)_p = 0$. Whence,

$l_{R_p}((N_{j+1}/N_j)_p) = l_{R_p}((N_{j+1}/L)_p) + l_{R_p}((L/N_j)_p) = 1 + 0 = 1$

.

Now in both cases it follows from Lemma 2.3 that N_{j+1} is a p-prime submodule of M such that $l_{R_p}((N_{j+1}/N_j)_p) = 1$. this completes the inductive step in the construction.

(iii) Let $\lambda_p(M) = n$ and $N_0 \subset \cdots \subset N_K$ be a saturated maximal chain of p-prime submodules of M. We show that $k = n-1$. By (i) we have $k \le n-1$. Since by assumption this cain is maximal it follows from the proof of (ii) that $l_{R_p}((M/N_k)_p) = 1$. Now suppose the contrary be true. Then the set

$E := \{N : N \text{ is a } p-\text{prime submodule of } M\}$,

has a unique minimal element $N' := \bigcap_{N \in E} N$ with respect to " $\subseteq$ ". So it follows from hypothesis that $N_0 = N'$. Also using (i) it follows from the proof of (ii) that $(N_0/pM)_p = 0$. Therefore,

$l_{R_p}((N_k/N_0)_p) = n - 1$.

Now suppose the contrary be true and $k < n-1$. Then we deduce that there is $0 \le j \le k$ ▯$- 1$, such that $l_{R_p}((N_{j+1}/N_j)_p) \ge 2$. Then there is $x \in N_{j+1}\backslash N_j$. By Lemma 2.3 we have $(N_j + Rx)/N_j \cong R/p$ and so $l_{R_p}(((N_j + Rx)/N_j))_p = 1$. Let $L := N_j + Rx$. Since N_{j+1}/L is the unique minimal element of the set

$\{N/L \ : N/L \text{ is a } p-\text{prime submodule of } M/L\}$,

again using (i) it follows from the proof of (ii) that $(N_{j+1}/L)_p = 0$. Thus we have

$2 \le l_{R_p}((N_{j+1}/N_j)_p) = l_{R_p}((N_{j+1}/L)_p) + l_{R_p}((L/N_j)_p) = 0 + 1 = 1$,

which is a contradiction. This completes the proof.

Now we need the following definitions.

Definition 2.5. Let R be a Noetherian ring and M be a finitely generated R-module. For each p-prime submodule N of M we define p-height of N as:

$p-ht(N) := \sup\{k \in N_0 : \exists\, N_0 \subset \cdots \subset N_K = N;\ \text{with } N_i \in Spec_R^p(M), \forall i\}$

where $\mathrm{Spec}_R^p(M)$ denotes to the set of all p-prime submodules of M as an R-module.

Definition 2.6. Let R be a Noetherian ring and M be a finitely generated R-module. For each p-prime submodule N of M we define height of N as:

$ht(N) := \sup\{k \in N_0 : \exists\, N_0 \subset \cdots \subset N_K = N;\ \text{with } N_i \in \mathrm{Spec}_R(M), \forall i\}$

,

where $\mathrm{Spec}_R(M)$ denotes to the set of all prime submodules of M as an R-module.

Definition 2.7. Let R be a Noetherian ring and M be a finitely generated R-module.

Then we define $\mathrm{dimSpec}_R(M)$ as:

$\mathrm{dimSpec}_R(M) := \sup\{ht(N) : N \in \mathrm{Spec}_R(M)\}$.

The following result is an immediately consequence of Theorem 2.4.

Corollary 2.8. Let R be a Noetherian ring and M be a finitely generated R-module and N be a p-prime submodule of M. Then

$p - ht(N) = l_{R_p}((N/pM)_p) = \dim_{R_p/pR_p}(N_p/pM_p)$.

Proof. Let $k := p - ht(N)$. Then there is saturated chain of p-prime submodules of M as $N_0 \subset \cdots \subset N_K = N$. By the proof of Theorem 2.4 this chain can be extended to a maximal saturated chain of p-prime submodules of M as

$N_0 \subset \cdots \subset N_K = N \subset \cdots \subset N_{n-1}$,

Where $\lambda_p(M) = n$. Then by the proof of Theorem 2.4 we have $(N_0/pM)_p = 0$ and $l_{R_p}((N_{i+1}/N_i)_p) = 1$, for each $0 \le i < n-2$. Now clearly the assertion holds.

As an application of Theorem 2.4 we prove the following.

Theorem 2.9. Let R be a Noetherian ring and M be a finitely generated R-module and N be a p-prime submodule of M. Then
$ht(N) \leq (\lambda_p(M))(\dim_{R_p}(M_p)) < \infty$.

Proof. Let $N_k \subset \cdots \subset N_0 = N$ be a chain of prime submodules of M, such that for each $0 \leq i \leq k$, N_i is p_i-prime, where $p_0 = p$. Then it easily follows from definition that
$p_k \subseteq \cdots \subseteq p_0 = p$.
Therefore, the set $\{p_i\}_{i=0}^{k}$ has at most $\dim_{R_p}(M_p)$ element. (Note that $p_i \in \mathrm{Supp}(M)$, for all $0 \leq i \leq k$). Let
$\{p_i\}_{i=0}^{k} = \{q_0 = p, \cdots, q_t\}$,
where $t \leq dim_{R_p}(M_p)$ and $p = q_0 \supset \cdots \supset q_t$. Let $A_j := Spec_R^{q_j}(M) \cap \{N_i\}_{i=0}^{k}$, for each $0 \leq j \leq t$. Then by Theorem 2.4 the set A_j has at most $\lambda_{q_j}(M)$ element. But $\lambda_{q_j}(M) \leq \lambda_p(M)$, because $q_j \subseteq p$. Therefore as
$\bigcup_{j=1}^{t} A_j = \{N_i\}_{i=0}^{k}$,
it follows that $k \leq t\lambda_p(M) \leq (\dim_{R_p}(M_p))\lambda_p(\mathrm{M})$. Which implies that
$\mathrm{ht(N)} \leq (\lambda_p(\mathrm{M}))(\dim_{R_p}(M_p)) < \infty$,
as required.

3. Prime avoidance Theorem

The results of this section which will be useful in the next section improve some well known results given in [8].

Proposition 3.1. Let R be ring and M be a non-zero R-module and N be a submodule of M. Let $p_1, \cdots, p_n$ be distinct prime ideals of R. Let for each $1 \leq i \leq n$, N_i be a p_i-prime submodule of M. If $N \subseteq \bigcup_{i=1}^{n} N_i$, then $N \subseteq N_j$ for some $1 \leq j \leq n$.

Proof. We do induction on n. The case $n = 2$ is easy. Now let $n \geq 3$ and the case $n - 1$ is settled. By definition for each $1 \leq i \leq n$ we have $p_i = (N_i :_R M)$. From the hypothesis $N \subseteq \bigcup_{i=1}^{n} N_i$ it follows that $N = \bigcup_{i=1}^{n}(N_i \cap N)$. Now let the contrary be true. Then $N \nsubseteq N_i$ and hence $(N_i \cap N) \neq N$, for any $1 \leq i \leq n$. Also from the inductive hypothesis it follows that $N \neq \bigcup_{i \in (\{1,\cdots,n\}) \setminus \{k\}}(N_i \cap N)$ for each $1 \leq k \leq n$ and so $(N_k \cap N) \nsubseteq \bigcup_{i \in (\{1,\cdots,n\}) \setminus \{k\}}(N_i \cap N)$. Let q be a minimal element of the set $\{p_1, \cdots, p_n\}$ with respect to " $\subseteq$ ". Then $p_i \nsubseteq q$ for each $p_i \in (\{p_1, \cdots, p_n\} \setminus \{q\})$. Without loss of generality we may assume that $q = p_n$. Let $J_i := (N_i :_R N)$, for all $i = 1, \cdots, n$. Then from the definition it follows that $p_i \subseteq J_i$, for all $i = 1, \cdots, n$. On the other hand for each $x \in N$ and $r \in R$, if $rx \in (N_i \cap N)$ and $x \notin (N_i \cap N)$, then $rx \in N_i$ and $x \notin N_i$. Therefore it follows from the definition that $r \in p_i$. So $rM \subseteq N_i$, and consequently, $rN \subseteq (N_i \cap N)$. As $(N_i \cap N) \neq N$ it follows that there exists an element $y \in (N \setminus (N_i \cap N))$. Now for each $s \in J_i$ we have $sy \in (N_i \cap N) \subseteq N_i$ and $y \notin N_i$. So it follows from the definition that $s \in p_i$. Therefore, $(N_i :_R N) = J_i = p_i = (N_i :_R M)$. But it is easy to see that $(N_i :_R N) = ((N_i \cap N) :_R N)$. Thus for each $1 \leq i \leq n$,
$N_i \cap N$ is p_i-prime submodule of N. Therefore without loss of generality we may assume that $N = M = \bigcup_{i=1}^{n} N_i$ and $N_n \nsubseteq \bigcup_{i=1}^{n-1} N_i$. Next let $T := \bigcap_{i=1}^{n} N_i$. Then it is not to see that for each $1 \leq i \leq n$, N_i/T is p_i-prime submodule of M/T and $M/T = \bigcup_{i=1}^{n} N_i/T$. Therefore, without loss of generality we may assume $M = \bigcup_{i=1}^{n} N_i$ and $\bigcap_{i=1}^{n} N_i = 0$ and $N_n \nsubseteq \bigcup_{i=1}^{n-1} N_i$. Then there is an exact sequence $0 \rightarrow M \rightarrow \bigoplus_{i=1}^{n} M/N_i$, which implies that $\bigcap_{i=1}^{n} p_i = \mathrm{Ann_R}(\bigoplus_{i=1}^{n} M/N_i) \subseteq \mathrm{Ann_R}(M)$. On the other hand for each $1 \leq i \leq n$ we have $\mathrm{Ann_R}(M) \subseteq (N_i :_R M) = p_i$. So $\mathrm{Ann_R}(M) \subseteq \bigcap_{i=1}^{n} p_i$. Hence $\mathrm{Ann_R}(M) = \bigcap_{i=1}^{n} p_i$. Now if we have $\bigcap_{i=1}^{n-1} N_i = 0$, then there is an exact sequence
$0 \rightarrow M \rightarrow \bigoplus_{i=1}^{n-1} M/N_i$ which implies that $\bigcap_{i=1}^{n-1} p_i = \mathrm{Ann_R}(\bigoplus_{i=1}^{n-1} M/N_i) \subseteq \mathrm{Ann_R}(M) = \bigcap_{i=1}^{n} p_i \subseteq p_n$. So $p_t \subseteq p_n$, for some $1 \leq t \leq n-1$, which is a contradiction. So $\bigcap_{i=1}^{n-1} N_i \neq 0$.Then there is an element $0 \neq a \in \bigcap_{i=1}^{n-1} N_i$. As $\bigcap_{i=1}^{n} N_i = 0$, it follows that $a \notin N_n$. On the other hand since $N_n \nsubseteq \bigcup_{i=1}^{n-1} N_i$, it follows that there is an element $b \in N_n$ such that $b \notin \bigcup_{i=1}^{n-1} N_i$. Now as $a + b \in \bigcup_{i=1}^{n} N_i$, it follows that $a + b \in N_k$ for some $1 \leq k \leq n$, which is a contradiction. This completes the inductive step.

Remark: Proposition 3.1 does not hold in general. For example let $p \geq 2$ be a prime number and $2 \leq n \in \mathbb{N}$. Let $R = \mathbb{Z}_p = \{\bar{0}, \bar{1}, \cdots, \overline{p-1}\}$ and $M = \bigoplus_{i=1}^{n} \mathbb{Z}_p$ Let
$\mathfrak{A} = \{N : N = Rx, \text{for some } 0 \neq x \in M\}$.
Then $\mathfrak{A}$ is a finite set that has at most 2^{p^n} element and for each $N \in \mathfrak{A}$, N is a $\{\bar{0}\}$-prime submodules of M such that $M \subseteq \bigcup_{N \in \mathfrak{A}} N$. But $M \nsubseteq N$ for any $N \in \mathfrak{A}$.

The following proposition is a generalization of [12, Ex. 16.8].

Proposition 3.2. Let R be a ring, M a non-zero R-module, N a submodule of M and $x \in M$. Let $p_1, \cdots, p_n$ be distinct prime ideals of R. Let for each $1 \leq i \leq n$, N_i be a p_i-prime submodule of M. If $N + Rx \nsubseteq \bigcup_{i=1}^{n} N_i$, then there exists $a \in N$ such that $a + x \notin \bigcup_{i=1}^{n} N_i$.

Proof. We use induction on n. Let $n = 1$. If $x \in N_1$ then $N \nsubseteq N_1$. So there is $a \in N \setminus N_1$ and it is easy to see that $a + x \notin N_1$. But if $x \notin N_1$, then by choosing $a = 0 \in N$ the assertion holds. Now suppose $n \geq 2$ and the case $n - 1$ is settled. Let q be a minimal element of the set $\{p_1, \cdots, p_n\}$ with respect to "$\subseteq$". Then $p_i \nsubseteq q$ for each $p_i \in (\{p_1, \cdots, p_n\} \setminus \{q\})$. Without loss of generality we may assume that $q = p_n$. Then it is easy to see that $\bigcap_{i=1}^{n-1} p_i \nsubseteq p_n$. By inductive hypothesis there is an element $b \in N$ such that $b + x \notin \bigcup_{i=1}^{n-1} N_i$. So the assertion hold for $a = b$, whenever $b + x \notin N_n$. So we may assume $b + x \in N_n$. Then we claim that $N \nsubseteq N_n$. Because, if $N \subseteq N_n$ then $x \in N_n$ and *so* $N + Rx \subseteq N_n \subseteq \bigcup_{i=1}^{n} N_i$, which is a contradiction. Therefore, there exists an element $c \in N \backslash N_n$. As $\bigcap_{i=1}^{n-1} p_i \nsubseteq p_n$ it follows that there exists an element $r \in (\bigcap_{i=1}^{n-1} p_i) \backslash p_n$. Then it easily follows from the definition of the p_n-prime submodule that $rc \notin N_n$. Moreover, since $r \in \bigcap_{i=1}^{n-1} p_i$ it follows from the definition that $rc \in \bigcap_{i=1}^{n-1} N_i$. Now it is easy to see that $rc + b + x \notin \bigcup_{i=1}^{n} N_i$. Therefore, the assertion hold for $a := rc + b \in N$. This completes the induction step.

Remark: Proposition 3.2 does not hold in general. For example let $p \geq 2$ be a prime number and $R = \mathbb{Z}_p = \{\bar{0}, \bar{1}, \cdots, \overline{p-1}\}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Let $N = (\bar{1}, \bar{0})\mathbb{Z}_p$, $x = (\bar{0}, \bar{1})$ and $N_i = (\bar{i}, \bar{1})\mathbb{Z}_p$, for $i = 0, \cdots, p-1$. Then N_i is $\{\bar{0}\}$-prime submodule of the R-module M, for all $i = 0, \cdots, p-1$. Also as $(\bar{1}, \bar{0}) \in N + Rx$ and $(\bar{1}, \bar{0}) \notin \bigcup_{i=0}^{p-1} N_i$, it follows that $N + Rx \nsubseteq \bigcup_{i=0}^{p-1} N_i$. But for any $a \in N$ we have $a + x \in \bigcup_{i=0}^{p-1} N_i$.

Now we give an other aspects of prime avoidance Theorem in different states.

Proposition 3.3. *Let R be a ring, M a non-zero R-module, N a submodule of M and $k \in \mathbb{N}$. Let for each $1 \leq i \leq k$, $n_i \in \mathbb{N}$ and for $1 \leq i \leq k$ and $1 \leq j \leq n_i$, the ideals $p_{i,j}$ be distinct elements of* Spec(R). *Let for each $1 \leq i \leq k$ and $1 \leq j \leq n_i$, $N_{i,j}$ be a $p_{i,j}$-prime submodule of M. Let for each $1 \leq i \leq k$, $N_i = \bigcap_{j=1}^{n_i} N_{i,j}$. If $N \subseteq \bigcup_{i=1}^{k} N_i$, then $N \subseteq N_t$ for some $1 \leq t \leq k$.*

Proof. Let the contrary be true. Then for each $1 \leq i \leq k$ we have $N \nsubseteq N_i$. Therefore there exists $1 \leq s_i \leq n_i$ such that $N \nsubseteq N_{i,s_i}$. But in this situation we have

$N \subseteq \bigcup_{i=1}^{k} N_i \subseteq \bigcup_{i=1}^{k} N_{i,s_i}$.

Consequently, it follows from proposition 3.1 that there is $1 \leq l \leq k$, such that $N \subseteq N_{l,s_l}$, which is a contradiction.

Proposition 3.4. *Let R be a ring, M a non-zero R-module, N a submodule of M, $x \in M$ and $k \in \mathbb{N}$. Let for each $1 \leq i \leq k$, $n_i \in \mathbb{N}$ and for $1 \leq i \leq k$ and $1 \leq j \leq n_i$, the ideals $p_{i,j}$ be distinct elements of* Spec(R). *Let for each $1 \leq i \leq k$ and $1 \leq j \leq n_i$, $N_{i,j}$ be a $p_{i,j}$-prime submodule of M. Let for each $1 \leq i \leq k$, $N_i = \bigcap_{j=1}^{n_i} N_{i,j}$. If $N + Rx \nsubseteq \bigcup_{i=1}^{k} N_i$ then there exists $a \in N$ such that $a + x \notin \bigcup_{i=1}^{k} N_i$.*

Proof. For each $1 \leq i \leq k$ we have $N + Rx \nsubseteq N_i$. Therefore there exists $1 \leq s_i \leq n_i$ such that $N + Rx \nsubseteq N_{i,s_i}$. But in this situation using proposition 3.1 we have

$N + Rx \nsubseteq \bigcup_{i=1}^{k} N_{i,s_i}$.

Consequently, it follows from proposition 3.2 that there is $a \in N$, such that $a + x \notin \bigcup_{i=1}^{k} N_{i,s_i}$. But since $\bigcup_{i=1}^{k} N_i \subseteq \bigcup_{i=1}^{k} N_{i,s_i}$, it follows that $a + x \notin \bigcup_{i=1}^{k} N_i$, as required.

Proposition 3.5. *Let R be a ring, I an ideal of R and $x \in R$. Let $J_1, \cdots J_n, (n \geq 1)$ be ideals of R such that for each $1 \leq i \leq n$ we have $Rad(J_i) = J_i$. If $I + Rx \nsubseteq \bigcup_{i=1}^{n} J_i$, then there exists an element $a \in I$ such that $a + x \notin \bigcup_{i=1}^{n} J_i$.*

Proof. For each $1 \leq i \leq n$ we have $I + Rx \nsubseteq J_i$. Therefore for each $1 \leq i \leq n$, since $J_i = \bigcap_{q \in V(J_i)} q$ it follows that there exists $p_i \in V(J_i)$ such that $I + Rx \nsubseteq p_i$. But in this situation we have $I + Rx \nsubseteq \bigcup_{i=1}^{n} p_i$. Consequently, it follows from [12, Ex. 16.8] that there is $a \in I$, such that $a + x \notin \bigcup_{i=1}^{n} p_i$. But since $\bigcup_{i=1}^{n} J_i \subseteq \bigcup_{i=1}^{n} p_i$, it follows that $a + x \notin \bigcup_{i=1}^{n} J_i$, as required.

Before bringing the next result we need the following well known lemma.

Lemma 3.6. *Let (R, m) be a commutative local ring such that R/m is infinite. Let M be an R-module and $N_1, \cdots, N_t$ be submodules of M such that $M = \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$, $M = N_j$.*

Proof. The assertion follows using NAK Lemma.

Proposition 3.7. *Let R be a commutative ring, M be an R-module and $N_1, \cdots, N_t$ be submodules of M such that $M = \bigcup_{i=1}^{t} N_i$. Then $\bigcap_{i=1}^{t} Supp\ M/N_i \subseteq Max(R)$.*

Proof. Suppose the contrary be true. Then there exists p$\in (\bigcap_{i=1}^{t} Supp\ M/N_i) \backslash Max(R)$. So R/p is an integral domain but not a field and therefore R_p/pR_p is infinite. By hypothesis and Proposition 3.6 there exists $1 \leq j \leq t$ such that $(M/N_j)_p = 0$ and so $p \notin Supp\ M/N_j$ which is a contradiction.

Corollary 3.8. *Let R be a commutative ring and $p \in Spec(R) \backslash Max(R)$. Let M be an R-module and $N_1, \cdots, N_t$ be p-prime submodules of M and N a submodule of M such that $N \subseteq \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$ such that $N \subseteq N_j$.*

Proof. Let for any $1 \leq j \leq t$, $N \nsubseteq N_j$. Then for all $1 \leq j \leq t$, we have $N \cap N_j \neq N$. Since $pM \subseteq N_j$, it follows that $pN \subseteq N_j$ and so $pN \subseteq N \cap N_j$. Hence $p \subseteq (N \cap N_j : N)$. On the other hand there exists $x \in N \backslash N \cap N_j$ and so $x \notin N_i$. Let $r \in (N_i \cap N : N)$. Then $rx \in N_i \cap N \subseteq N_i$ and $x \notin N_i$, so $r \in (N_i : M) = p$. Consequently $(N_i \cap N : N) \subseteq p$ and so $(N_i \cap N : N) = p$. Now it is easy to show that $N_i \cap N$ is a p-prime submodule of N. Since $N \subseteq \bigcup_{i=1}^{t} N_i$ it follows that $N = \bigcup_{i=1}^{t} (N \cap N_i)$. But in this case $p \in \bigcap_{i=1}^{t} Supp\ (N/N_i \cap N)$. Since $p \in Spec(R) \backslash Max(R)$ this is impossible by Proposition 3.7.

Proposition 3.9. *Let R be a commutative ring and $p \in Spec(R)$ such that R/p infinite. Let M be an R-module and $N_1, \cdots, N_t$ be p-prime submodules of M and N a submodule of M such that $N \subseteq \bigcup_{i=1}^{t} N_i$. Then there exists $1 \leq j \leq t$ such that $N \subseteq N_j$.*

Proof. If $p \notin Max(R)$, the assertion follows from Corollary 3.8. So let $p \in Max(R)$ and for all $1 \leq i \leq t$, we have $N \nsubseteq Ni$. Hence for any $1 \leq j \leq t$, there exists $x_j \in N \backslash N_j$. Set $N' = (x_1, \cdots, x_t) \subseteq N$ and so we have $N'/pN' = \bigcup_{i=1}^{t} ((N' \cap N_i) + pN')/pN'$. Since R/p is infinite, there exists $1 \le j \le t$ such that $N'/pN' = ((N' \cap N_j) + pN')/pN'$. This implies that $N' = (N' \cap N_j) + pN' \subseteq pM + N_j = N_j$. Hence $N' \subseteq N_j$ which is a contradiction.

Proposition 3.10 *Let R be a commutative ring and $p \in Spec(R)$ such that R/p infinite. Let M be an R-module and $N_1, \cdots, N_t$ be p-prime submodules of M and N a submodule of M. Let $x \in M$ such that $N + Rx \nsubseteq \cup_{i=1}^{t} N_i$. Then there exists $a \in N$ such that $a + x \notin \cup_{i=1}^{t} N_i$.*

Proof. It is certainly true for *t* = 1. Let *t* > 1 and the result has been proved for $t-1$. If $N \subseteq \cup_{i=1}^{t} N_i$ then by Proposition 3.9 there exists $1 \le j \le t$, such that $N \subseteq N_j$. Without loss of generality we may assume that *j* = *t*. By induction hypothesis there exists $b \in N$ such that $b + x \notin \cup_{i=1}^{t-1} N_i$. Since $b + x \notin N_t$ it follows that $b + x \notin \cup_{i=1}^{t} N_i$ and so the assertion follows. Now suppose that $N \nsubseteq \cup_{i=1}^{t} N_i$, then there exists $c \in N \setminus \cup_{i=1}^{t} N_i$. In this case if $x \notin \cup_{i=1}^{t} N_i$ we set *a* = 0 and if $x \in \cap_{i=1}^{t} N_i$ then we set *a* = *c*. Now suppose that the above conditions are not true. We may assume that there exists $1 \le k \le t-1$ such that $x \in \cap_{i=1}^{k} N_i$ and $x \notin \cup_{i=k+1}^{t} N_i$. Since R/p is infinite, so there exist $t-k+1$ non-zero distinct elements in R/p such as $s_1 + p, \cdots, s_{t-k+1} + p$. Set $A = \{s_i c + x \mid i = 1, \cdots t-k+1\}$. If there exists an element $s_i c + x$ *in* A such that $s_i c + x \notin \cup_{i=1}^{t} N_i$ then the proof is complete. Otherwise, for each $1 \le l \le t-k+1$, there is $1 \le j \le t$ such that $s_l c + x \in N_j$. If $1 \le j \le k$ then $s_l \in p$ and so $s_l + p = p$ which is a contradiction. So $k+1 \le j \le t$ and hence $A \subseteq \cup_{i=k+1}^{t} N_i$. Whence, according to the Dirichlet drawer principle, there exists $k+1 \le j \le t$ and $1 \le l_1 < l_2 \le t-k+1$ such that $s_{l_1} c + x$ and $s_{l_2} c + x$ belong to N_j. Therefore $s_{l_1} + p = s_{l_2} + p$ which is a contradiction.

4. Minimal prime submodules

The following lemma is needed in the proof of the first main result of this section. Note that in the sequel for any submodule *B* of an *R*-module *M*, the set of all minimal prime submodules of *M* over *B* is denoted by Min(*B*). Moreover, we denote Min(0) by Min(*M*). Also, *V* (*B*) is defined as follows:

$V(B) = \{N \in Spec_R(M) : N \supseteq B\}$.

Lemma 4.1. *Let R be a commutative ring and $p, q \in Spec(R)$. Let M be an R-module and $N_1, N_2 \in Min\ M$ be respectively p-prime and q-prime submodules. Then $N_1 \neq N_2$ if and only if $p \neq q$.*

Proof. If $p \neq q$ then obviously $N_1 \neq N_2$. Conversely, Let $N_1 \neq N_2$ but $p = q$. Since $L_1 = \cap_{L \in Spec_R^p(M)} L$ and $L_2 = \cap_{L \in Spec_R^q(M)} L$ it follows that *L*1 = *L*2 which is a contradiction.

Definition 4.2. Let *M* be an *R*-module and *B* be a submodule of *M*. Set

$D(B) := \{N \in \mathrm{Min}(B) : N \text{ is not finitely generated } R-module\}$

The minimal prime submodules of an *R-module M* has been studied in [16], for example see [16, Theorem 2.1]. In the next theorem we present a new conditions that an *R-module M* has only a finite number of minimal prime submodules, whenever *R* is a Noetherian ring, which is a generalization of [2, Theorem 2.1].

Theorem 4.3. Let R be a Noetherian ring, M be an R-module and B be a submodule of M. Then the following statements are equivalent:

(1) Min(B) is finite.

(2) For every $\mathfrak{P} \in \mathrm{Min}(B)$ there exists a finitely generated submodule $K_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $|V(K_{\mathfrak{P}}) \cap \mathrm{Min}(B)| < \infty$.

(3) For every $\mathfrak{P} \in \mathrm{Min}(B)$ there exists a finitely generated submodule $N_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $V(N_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$.

(4) For every $\mathfrak{P} \in \mathrm{Min}(B)$, $\mathfrak{P} \nsubseteq \cup_{L \in Min(B) \setminus \{\mathfrak{P}\}} L$.

(5) For every $\mathfrak{P} \in \mathrm{Min}(B)$ there exists an element $x_{\mathfrak{P}} \in \mathfrak{P}$ of $\mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$.

(6) For every $\mathfrak{P} \in \mathrm{D}(B)$, $\mathfrak{P} \nsubseteq \cup_{L \in Min(B) \setminus \{\mathfrak{P}\}} L$.

(7) For every $\mathfrak{P} \in \mathrm{D}(B)$ there exists an element $x_{\mathfrak{P}} \in \mathfrak{P}$ of $\mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$.

(8) For every $\mathfrak{P} \in \mathrm{D}(B)$ there exists a finitely generated submodule $K_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $|V(K_{\mathfrak{P}}) \cap \mathrm{Min}(B)| < \infty$.

(9) For every $\mathfrak{P} \in \mathrm{D}(B)$ there exists a finitely generated submodule $N_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $V(N_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$.

Proof. Without loss of generality, we may assume that *B* = 0, $Spec_R(M) \neq \emptyset$ and consequently $\mathrm{Min}(M) \neq \emptyset$.

$(1) \Rightarrow (2)$ Since Min(M) is finite, by Lemma 4.1 and Proposition 3.1, for every $\mathfrak{P} \in \mathrm{Min}(M)$, $\mathfrak{P} \nsubseteq \cup_{L \in Min(M) \setminus \{\mathfrak{P}\}} L$ and there exists $x \in \mathfrak{P} \setminus \cup_{L \in Min(M) \setminus \{\mathfrak{P}\}} L$. Set $K_{\mathfrak{P}} = Rx$. Then $K_{\mathfrak{P}}$ is finitely generated and set $V(K_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$ is finite.

$(2) \Rightarrow (3)$ Let $\mathfrak{P} \in \mathrm{Min}(M)$ and $V(K_{\mathfrak{P}}) \cap \mathrm{Min}(M) = \{\mathfrak{P}, \mathfrak{P}_2, \cdots, \mathfrak{P}_n\}$. Using Lemma 4.1 and Proposition 3.1 we can find an element $x \in \mathfrak{P} \setminus \cup_{i=2}^{n} \mathfrak{P}_i$. Let $N_{\mathfrak{P}} := K_{\mathfrak{P}} + Rx$. Then $N_{\mathfrak{P}}$ is finitely generated and $V(N_{\mathfrak{P}}) \cap \mathrm{Min}(M) = \{\mathfrak{P}\}$.

$(3) \Rightarrow (1)$ Suppose the contrary be true. Then the set Min(*M*) is infinite. Let

$A := \{p \in \mathrm{Spec}(R) : Spec_R^p(M) \cap \mathrm{Min}(M) \neq \emptyset\}$

$E := \{N \le M : N \text{ is finitely generated and } V(N) \cap \mathrm{Min}(M) \neq \emptyset \text{ is a finite set.}\}$

$F := \{L \le M : \forall N \in E, N \nsubseteq L\}$

We show that there exists a maximal element *K* of *F* such that $(K :_R M)$ is a prime ideal. Since Min(*M*) is infinite, so the zero submodule of *M* belong to the *F* and therefore by Zorn's Lemma *F* has a maximal element. Let *L* be a maximal element of *F*. If $(L :_R M)$ be a prime ideal, we are through. If not, then it is clear that $(L :_R M) \neq R$. Let $q_1 \in Ass_R(R/(L :_R M))$. By the definition there exists

$r \in R \backslash (L :_R M)$ such that $q_1 = ((L :_R M) : r)$ and therefore $q_1 rM \subseteq L$. Since $r \notin (L :_R M)$, it follows that there exists an element $x \in M$ such that $rx \notin L$. Now there exists $N \in E$ such that $N \subseteq L + Rrx$. In particular,
$q_1 N \subseteq L + q_1 rx \subseteq L + q_1 rM \subseteq L$.
Since $q_1 N$ is finitely generated, so $|V(q_1 N) \cap \mathrm{Min}(M)| = \infty$. But in this case for all $\mathfrak{P} \in (V(q_1 N) \cap \mathrm{Min}(M) \backslash V(N) \cap \mathrm{Min}(M))$, we have $q_1 N \subseteq \mathfrak{P}$ and $N \nsubseteq \mathfrak{P}$. Now if $\mathfrak{P}$ be a p-Prime submodule, then $q_1 \subseteq p$ and so $|V(q_1) \cap A| = \infty$. Hence $|V(q_1 N) \cap \mathrm{Min}(M)| = \infty$. So for all $N \in E$, we have $N \nsubseteq q_1 M$ and therefore $q_1 M \in F$. Let
$U := \{q \ \in V(q_1) : qM \in F\}$.
Since R is Noetherian it follows that U has a maximal element, say q_2. $q_2 M \subseteq H$, for some maximal element H of F. We claim that $(H :_R M)$ is a prime ideal of R. If not, according to the above argument, there exists $q_3 \in Ass_R(R/(H :_R M))$ such that $q_3 M \in F$ and $q_2 \subsetneq (H :_R M) \subseteq q_3$. By choosing of q_2, we must have $q_2 = q_3$, which is a contradiction. Therefore $(H :_R M) = q_2$ is a prime ideal. Now we show that H is a q_2-prime submodule. Otherwise there exist $x \in M \backslash H$ and $r \in R \backslash q_2$, such that $rx \in H$.So $r \in Z_R(M/H) = \bigcup_{q \in Ass_R(M/H)} q$ and hence there exists $q' \in Ass_R(M/H)$ such that $r \in q'$. Consequently, $q_2 \subset q'$. On the other hand by definition $q' = (H :_R y)$ for some $y \in M \backslash H$. Since $H \subset H + Ry$, it follows that there exists $N \in E$ such that $N \subseteq H + Ry$ and so $q'N \subseteq H$. According to the above argument, $|V(q'M) \cap \mathrm{Min}(M)| = \infty$ which implies $q'M \in F$. Finally, we have $q_2 = (H :_R M) \subset q'$, which is a contradiction withthe choosing of q_2. Therefore H is a q_2-prime submodule of M. Whence, H contains a minimal prime submodule of M such as $\mathfrak{P}$. By assumption there exists a submodule $N_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $N_{\mathfrak{P}} \subseteq \mathfrak{P} \subseteq H$ and $N_{\mathfrak{P}} \in E$, which is a contradiction. Therefore, Min(M)is a finite set.
Now the proof of $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ is complete.
$(1) \Rightarrow (4)$ Follows from Lemma 4.1 and Proposition 3.1.
$(4) \Rightarrow (1) \Leftrightarrow (5)$ Since $(5) \Leftrightarrow (4) \Rightarrow (3)$ is clear so we have $(1) \Leftrightarrow (4) \Leftrightarrow (5)$.
Now we have the following: $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$.
$(4) \Rightarrow (6)$ Is clear.
$(6) \Rightarrow (3)$ Since for every $\mathfrak{P} \in D(0)$, $\mathfrak{P} \nsubseteq \bigcup_{L \in Min(M) \backslash \{\mathfrak{P}\}} L$, it follows that thereexists $x_{\mathfrak{P}} \in \mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$. On the other hand for all $\mathfrak{P} \in (Min(M) \backslash D(0))$, we have $V(\mathfrak{P}) \cap \mathrm{Min}(B) = \{\mathfrak{P}\}$, where $\mathfrak{P}$ is finitely generated. So the assertion follows.
$(6) \Leftrightarrow (7)$ and $(1) \Rightarrow (8), (9)$ are clear.
$(8), (9) \Rightarrow (3)$ Follow by a similar arguments as in $(6) \Rightarrow (3)$.
The following results follow from Theorem 4.3.
Corollary 4.4. *Let R be a Noetherian ring, M an R-module and B be a proper submodule of M. Then* Min(B) *is infinite if and only if there exists* $\mathfrak{P} \in D(B)$ *such that* $\mathfrak{P} \subseteq \bigcup_{L \in (Min(M) \backslash \{\mathfrak{P}\})} L$.
Proof. Follows immediately from Theorem 4.3.
Corollary 4.5. *Let R be a Noetherian ring, M an R-module and B be a proper submodule of M such that any minimal prime submodule over B is finitely generated. Then* Min(B) *is finite.*
Proof. Follows immediately from Theorem 4.3.
Definition 4.6. Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module and N be a proper submodule of M. Then *the radical of N* is defined as:
$\mathrm{Rad}(N) = \bigcap_{L \in \mathrm{Min}\, N} L$.
Before bringing the next definition, recall that for any ideal I of a Noetherian ring, the *arithmetic rank* of I, denoted by ara(I), is the least number of elements of I required to generate an ideal which has the same radical as I, i.e.,
$\mathrm{ara}(I) := \min\{n \in \mathbb{N}_0 : \exists\, x_1, \cdots, x_n \in I \text{ with } \mathrm{Rad}((x_1, \cdots, x_n)) = \mathrm{Rad}(I)\}$
Definition 4.7. Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module and N be a proper submodule of M.
We define the *arithmetic rank* of N, as:
$\mathrm{ara}(N) := \min\{n \in \mathbb{N}_0 : \exists\, x_1, \cdots, x_n \in N \text{ with } \mathrm{Rad}((x_1, \cdots, x_n)) = \mathrm{Rad}(N)\}$
The next theorem is a generalization of [14, Theorem 2.7].
Theorem 4.8. Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module and N be a proper submodule of M. *Then* $ara(N) \leq dim\, Spec_R(M) + 1$.
Proof. Let $d := dim\, Spec_R(M)$. We may assume that d is finite. Now, suppose, to the contrary, that $ara(N) > d + 1$. Let $n := ara(N)$. Since $n > d + 1 \geq 1$ it follows from the definition that there exist elements $x_1, \cdots, x_n$ in N such that $Rad(N) = Rad((x_1, \cdots, x_n))$.As $n > 0$ it follows that $\mathrm{Min}(0) \backslash V(N) \neq \emptyset$. Therefore it follows from Lemma 4.1and proposition 3.1 that $N \nsubseteq \bigcup_{L \in \mathrm{Min}(0) \backslash V(N)} L$. Therefore $(x_1, \cdots, x_n) \nsubseteq \bigcup_{L \in \mathrm{Min}(0) \backslash V(N)} L$, and so by Proposition 3.2 there is $a_1 \in (x_2, \cdots, x_n)$ such that
$x_1 + a_1 \notin \bigcup_{L \in \mathrm{Min}(0) \backslash V(N)} L$.
Let $y_1 := x_1 + a_1$. Then $y_1 \in N$ and $\mathrm{Rad}(N) = \mathrm{Rad}((y_1, x_2, \cdots, x_n))$. We shall construct the sequence $y_1, \cdots, y_{n-1} \in N$ such that $\mathrm{Rad}(N) = \mathrm{Rad}((y_1 \cdots y_{n-1}, x_n))$ and $y_j \notin \bigcup_{L \in \mathrm{Min}((y_1, \cdots, y_{j-1})) \backslash V(N)} L$, for each $1 \leq j \leq n - 1$, by an inductive process. To do this end, assume that $1 \leq k < n - 1$, and that we have already constructed elements $y_1, \cdots, y_k$ such that
$Rad(N) = Rad((y_1, \cdots, y_k, x_{k+1}, \cdots, x_n))$.

We show how to construct y_{k+1}. To do this, as $k < n - 1$ it follows that
$\mathrm{Min}(y_1, \cdots, y_k) \backslash V(N) \neq \emptyset$.
Therefore it follows from Lemma 4.1 and proposition 3.1 that
$N \nsubseteq \bigcup_{L \in \mathrm{Min}(y_1, \cdots, y_k) \backslash V(N)} L$.
Therefore
$(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n) \nsubseteq \bigcup_{L \in \mathrm{Min}(y_1, \cdots, y_k) \backslash V(N)} L$, and so

by Proposition 3.2 there is $a_{k+1} \in (y_1, \cdots, y_k, x_{k+2}, \cdots, x_n)$ such that
$x_{k+1} + a_{k+1} \notin \bigcup_{L \in \mathrm{Min}(y_1,\cdots,y_k)\setminus V(N)} L$.
Let $y_{k+1} := x_{k+1} + a_{k+1}$. Then $y_{k+1} \in N$ and $Rad(N) = Rad((y_1, \cdots, y_k, y_{k+1}, x_{k+2}, \cdots, x_n))$.This completes the inductive step in the construction. Now it is easy to see that $\mathrm{Min}(y_1, \cdots, y_{n-1})\setminus V(N) \neq \emptyset$. Also using an induction argument we can deduce that for any $1 \leq j \leq n - 1$ and any $L \in \mathrm{Min}(y_1, \cdots, y_j)\setminus V(N)$ we have $ht(L) \geq j$. Consequently, since there exists a prime submodule L of M in which $L \in \mathrm{Min}(y_1, \cdots, y_{n-1})\setminus V(N)$it follows that $n - 1 \leq ht(L) \leq dimSpec_R(M) = d$. Which implies that $n \leq d + 1$, as required.

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