Takestan Institute of Higher Education

Available online at http://UCTjournals.com

UCT Journal of Research in Science, Engineering and Technology

UCT. J. Resea. Scien. Engineer.Techno. (UJRSET)

Volume 3,Issue2 01-07 (2015) ISSN 2382-977X

Existence and uniqueness theorem for fuzzy fractional integral equations with the fuzzy caputo fractional derivative by using Adomians decomposition method (ADM)

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ABSTRACT

We present on existence and uniqueness theorem for integral equation of fractional order in volving fuzzy set value mayping of a real variable whose values are normal , convex , upper , semicontinuous , and compactly sufforted fuzzy sets in IR^n ir , we establish condition for a class of initial value froblem for impulsive frac tional antegrab inclusion involving the caputo fractional derivative. The Adomians decomposition method and the homotopy Perturbation method are two powerful method which Consider the approximate Solution of a nonlinear Equation as an infinite Series usually Converging to the accurate solution , This paper introduces the homotopy perturbation method for overcoming completely the disadvantage, The solution procedure is very effective and straight forward. That two method are equivalent in solving nonlinear equations.

Original Article:

Received:20 July
Accepted :20 August
Published :23 September

Keywords:
Fuzzy Fractional differential equations, Caputo fractional derivative, Fuzzy Fractional Solution, Existence and uniqueness

Introduction

Dubios and prade [1]introduced the concept of integration of fuzzy functions

Alternative affrouches were later suyyested by goetsch and voxman [3] , kaleve [4] Nanda [5] and others , while Goetschol and voxman preferred a Riemann and caputo integral type approach , Kaleva choseto define the integral of fuzzy function for more information about integralin of fuzzy function and fuzzy integral equations for instance , see [6],[7], and reference therein . We denote the set of all real number by R,and the set of all fuzzy numbers on R is indicated by R_F.Now , the fractional calculus topic is enjoying is growing interest among scientists and engineers , see [8,9,10,11,12,13]this equation takes the form. . In this paper, we propose Riemann-liouville differentiability by using Hukuhara difference so-called Riemann-liouville H-differentiability. [7, 13], the successful application of fractional integration Equations (FIES) in mo de ling such as viscose lastic material [1], control [2], signal processing [3] and etc.

This paper is organized as follows: In Section [2], we recall some Well-known definitions of fuzzy number and express some needed concepts. In Sect [3], Riemann-Liouville H-differentiability is given and Caputo of (FFIEs) are considered under Caputo H-differentiability Fuzzy fractiond intial value problem. In Sect[4]. Existence and uniqueness with of the caputo fuzzy fractional derivative by using Adomians decomposition method(ADM). In Sect.[5] some examples are solved FFIES Under Caputo GH-differentiability.

Consider the following Fredholm integral equation

$$f(x) = g(x) + \mu \int_a^b k(x,t) f(t) \quad a < x \leq b \tag{1}$$

where g and k are known functions and f is to be determined. The

Adomians decomposition method consists of representing f as a series

$$f(x) = \sum_{n=0}^{\infty} u_n(x) \tag{2}$$

Now by substituting (1) in (2), we will have

$$\sum_{n=0}^{\infty} u_n(x) = g(x) + \mu \int_a^b k(x,t) \sum_{n=0}^{\infty} u_n(t)\, dt \tag{3}$$

Note that,Adomians decomposition method uses the recursive relations

$$U_{n+1} = \mu \int_a^b k(x,t) \sum_{n=1}^{\infty} u_n(t)\,dt, \qquad U_0(x) = g(x)$$
(4)
where
$\mu > 0, k(x,t)$
is an arbitrary kernel function over the square
$a \le s,t \le b$
and
$f(t), a \le t \le b$
is a function of t : $a \le t \le b$. If f $f(t)$ is a crisp function then the solutions
of Eq. (1) are crisp as well. However, if $f(t)$ is a fuzzy function these equations
may only possess fuzzy solutions. Sufficient conditions for the existence of
a unique solution to the fuzzy Fredholm integral equation of the second kind,
i.e. to Eq. (2) where $f(t)$ is a fuzzy function, are given in [6] let
$(f_-(t,r), f^-(t,r))$
and
are parametric
form of $f(t)$ and $u(t)$, respectively then, parametric form of FFIE-2 is as
follows:, $0 \le r \le 1$

$$u_-(t,r) = f_-(t,r) + \mu \int_a^b v_1(s,t,u_-(s,r),u^-(s,r)$$

$$u^-(t,r) = f^-(t,r) + \mu \int_a^b v_2(s,t,u_-(s,r),u^-(s,r)ds$$

$0 \le r \le 1$
we explain Adomian method as a numerical algorithm for approximating solution of this system of linear integral equations in crisp case
then, we find approximate solutions for $u(t)$ and $f(t)$ for each $a \le t \le b$ and
$0 \le r \le 1$

$$x(t) = g(t) + \frac{1}{\Gamma(\beta)} \int_0^r \frac{f(t.y(t))}{(r-t)^{1-\beta}}dt \qquad r \in [.,T] \qquad (1-1)$$

Where g: $[.,T] \to E^n$ and $f = [0,T] \times E^n \to E^n$
2-.preliminaries
De finition 2-1.
Let g∈L(a,b) , .≤a<b<∞ , and let β>0
Bea real number the fractional integral of order β of Riemann – liouville type define by (see [12],[13])

$$I^\beta g(t) = \frac{1}{\Gamma p(\beta)} \int_0^r \frac{g(t)}{(r-t)^{1-\beta}} dt \qquad (2.1.1)$$

Rewrite Eq(1.1)in the form
X(t)=g(t)+ $I^\beta f$ (t,y(t)) (2-1-2) t≥0
Where I^β is the standard Riemann –liouville fractional integral operator .

In this paper , we ∫rore an existenee and uniqueness theorem of a solution to the fuzzy integral equation (2-1-2) the method of successive approximation is the main toob in our analysis .
Definition 2-2 [14,15]
For a function g given on the interval [a,b] .
The caputo fractionaL order derivative of g is defined by

$$({}^c_a p^\beta g(t) = \frac{1}{\Gamma(n-\beta)} \int_a^r (r-t)^{n-\beta-1} g(t)^n dt \qquad (2.2.1)$$

Where r=$[R(\beta)]$ +1
Sufficient condition for the fractional differential and integrals to exist are given in [14]
Definition 2-3
The fractional order integral of the function g∈($\acute{L}[a,b], R^+$)of order $\beta \in R^+$ is defined by

$$I^\beta_{a+} g(t) = \int_0^r \frac{(r-t)}{\Gamma(\beta)} g(t)dt \qquad (2.5.1)$$

Where Γ is the gamma funetion where
α=. , we write $I^\beta g(t)$=g(t)×Q_β(t)
Where
$Q_\beta(t) = \frac{r^{\beta-1}}{\Gamma(\beta)}$ for $r > 0$, and $Q_\beta(t) = 0$ for r≤0 and $Q_\beta \to \delta(t)$
as $\to 0$, where δ is the delta function definition (2-4) caputo fractional derivatrve of order β (0<β<1) for u(t): R →R is defined as

$$D^\beta_c u(t) = \frac{1}{\Gamma(1-\beta)} \int_.^r (r-t)^{-\beta} (\frac{d}{dt}(u(t))dt \qquad (2.4.1)$$

Definition 2-4
initial value problem with fractional derivative

$$D^r_c y(t) = f(x,t) \qquad t \in [t.,T] \qquad (2.5.1)$$

X.($t_.$)=X.∈$X\,E^n$ $t_. \in [.,T]$
$[y(t)]^\beta = [y_1(t,\beta), y_2(t,\beta)]$
3-Fuzzy fractiond intial value problem
Let
y:$[0,T] \to E^n$ *be a fuzzy function of a* Cris ∫ raviable for
$[y(t)]^\beta = [y_1(t,\beta), y_2(t,\beta)]$
We have $[\acute{y}(t)]^\beta = [\acute{y}_1(t,\beta), \acute{y}_2(t,\beta)]$
Since r-t>. then $D^r_c y(t)$ can be defined levelwise as

$$\left[D^\beta_c y(t)\right]^\alpha = \left[D^\beta_c y_1(t,\beta), D^\beta_c y_2(t,\beta)\right] \qquad (3-1)$$

$$D_c^{\beta} y_1(t,\alpha) = \frac{1}{\Gamma(1-\beta)} \int_0^r (r-t)^{-\beta} \acute{y}_1(t,\alpha)dt$$

$$D_c^{\beta} y_2(t,\alpha) = \frac{1}{\Gamma(1-\beta)} \int_0^r (r-t)^{-\beta} \acute{y}_2(t,\alpha)dt \quad (3-2)$$

Theorem 3-1

Let .<β<1 and y: $[a,b] \to E^n$ be a fuzzy Function with

$$[y(t)]^{\alpha} = [y_1(t,\alpha), y_2(t,\alpha)]$$

i)if y has franctional derivative of type

$$\lim_{h\to 0^+} \frac{f(t.+h)+f(t.)}{h} = \lim_{h\to 0^-} \frac{f(t.)-f(t.-h)}{h}$$

then y_1 , y_2 , have fractional derivative and

$$\left[D_c^{\beta} y(t)\right]^{\alpha} = \left[D_c^{\beta} y_1(t,\alpha), D_c^{\beta} y_2(t,\alpha)\right] \quad (3-1-2)$$

ii)if y has fractional derivative of type

$$\lim_{h\to .^+} \frac{f(t.+h)+f(t.)}{h} = \lim_{h\to .^-} \frac{f(t.)-f(t.-h)}{h}$$

Them y_1,y_2 , have fractional derivative and

$$[\overset{\beta}{\underset{C}{D}} y(t)]^{\alpha} = [\overset{\beta}{\underset{C}{D}} y_2(t\alpha) \ , \overset{\beta}{\underset{C}{D}} y_1(t_1\alpha)] \quad (3-1-3)$$

Proof:

We Prove part i and the same proof can be used for Part be used for part ,Sinee $0 \le \beta <$ and

$[y(t)]^{\alpha} = [y_1(t_1,\alpha), y_2(t_1,\alpha)]$ then

$$[y'(t)]^{\alpha} = [y'_1(t,\alpha), y'_2(t,\alpha)] \quad (3-1-4)$$

$$[(r-t)^{-\beta} y'(t)]^{\alpha} = [(r-t)^{-\beta} y'_1(t,\alpha), (r-t)^{-\beta} y'_2(t,\alpha)]$$

Sinie $0 < \beta < 1$ then $\Gamma(1-\beta) > 0$, there fore

$$\frac{1}{\Gamma(1-\beta)} \int_0^r (r-t)^{-\beta} y'(t)dt]^{\alpha} = \quad (3-1-5)$$

$$\frac{1}{\Gamma(1-\beta)} \int_0^r (r-t)^{-\beta} y'_1(t\alpha)dt, \frac{1}{\Gamma(1-\beta)} \int_0^r (r-t)^{-\beta} y'_2(t,\alpha)dt]$$

$$\delta o[\overset{\beta}{\underset{C}{D}} y(t)]^{\alpha} = [\overset{\beta}{\underset{C}{D}} y_1(t,\alpha), \overset{\beta}{\underset{C}{D}} y_2(t,\alpha)]$$

Definifon 3-1 *[2]*

The generalized Hukuhara derivative of a fuzzy -valued function $f:(a,b) \to R_F$ *at* t_o *defined as*

$$f'_{gH}(t_o) = \lim_{h\to 0} \frac{f(t_o+h)!_{gH} f(t_o)}{h} = \lim_{h\to 0} \frac{f(t_o)!_{gH} f(t_o-h)}{h}$$

(3-1-6)

$f'_{gH}(r,t_o) \in R_F$ we say that f is generalized Hukuhara differentiable (gH)-differentiable at t_o Also we say that f is $[(i)-gH]$-differentiable at t_o if

$$f'_{GH}(r,t_o) = [f'_-(r,t_o), f'^-(r,t_o)], r \in [0,1]$$

(3-1-7)

and that f is $[(ii)-gH]$-differentiable at t_0

$$f'_{gH}(r,t_o) = [f'^-(r,t_o), f'_-(r,t_o)], r \in [0,1]$$

Definilion3-2

Let

$$R(\beta) \ge 0 \ , \ n = [R(\beta)] + 1 \quad (3-1-6)$$

also y(x) $\in C^n[a,b]$, Let n be given (3-1-7)

$n = [R(\beta)] + 1$ for $\beta \notin N_0$, $n = \beta$ for $\beta \in N_0$ (3-1-1)

$$(\overset{c\beta}{\underset{a^+}{D}} y)(x,\alpha) = [\overset{C\beta}{\underset{a^+}{D}} \underline{y}(x,\alpha), \overset{C\beta}{\underset{a^+}{D}} \overline{y}(x,\alpha)] \quad (3-1-8)$$

$$(\overset{c\beta}{\underset{a^+}{D}} \underline{y})(x,\alpha) = \frac{1}{\Gamma(n-\beta)} \int_a^x \underline{y}^{(n)} \quad (t,\alpha) / (x-t)^{\beta-n+1} dt \quad (3-1-9)$$

$$(\overset{c\beta}{\underset{a^+}{D}} \overline{y})(x,\alpha) = \frac{1}{\Gamma(n-\beta)} \int_a^x \overline{y}^{(n)} \quad (t,\alpha) / (x-t)^{\beta-n+1} dt \quad (3-1-10)$$

4- Existence and uniqueness FFIES with of the caputo fuzzy fractional derivative

In this section we consider the ca puto fuzey differential equation of order β >0

$$[(\overset{c\ \beta}{\underset{a^+}{D}} y)]^{\alpha}(x) = f[x, y(x,\alpha)] \quad \beta > 0 \quad , \quad a \le x \le b \quad (4-1)$$

Involving the fuzzy capu to fractional derivative $\overset{c\beta}{\underset{a^+}{D}} y(x,\alpha)$, defined in (4-1)

On a finite the initial conditions

$$[y^{(k)}]^{\alpha}(a) = b_k \quad (b_k \in R \ , \ k = 0,1,..., / n-1 \quad n = -[-\beta]) \quad (2-)$$

Definition 4-1

Let [a,b] be afinite interval of the real line R , and Let

$$[\overset{c\beta}{\underset{a^+}{D}} y(t)]^{\alpha}(x) = (\overset{C\beta}{\underset{a^+}{D}} y)(x,\alpha)$$

$$and \ [\overset{c\beta}{\underset{b^-}{D}} y(t)]^{\alpha}(x) = (\overset{C\beta}{\underset{b^-}{D}} y)(x,\alpha) \ \ be \ the \quad (4-1-1)$$

Lemma 4-1

Riemann-liouvill fractional derivatives of order $\beta \in C(R(\beta) \ge 0)$ *de fined by*

$$(\overset{C\beta}{\underset{a^+}{D}} \underline{y})(x,\alpha) = (\frac{d}{d_x})^n (\overset{n-\beta}{\underset{a^+}{I}} \underline{y})(x,\alpha) = \quad (x > a)$$

$$= \frac{1}{\Gamma(n-\beta)} (\frac{d}{dx})^n \int_a^x \frac{\underline{y}(t,\alpha)}{(x-t)\beta-n+1} dt \quad (4-1-2)$$

$$(\overset{C\beta}{\underset{a^+}{D}} \overline{y})(x,\alpha) = (\frac{d}{d_x})^n (\overset{n-\beta}{\underset{a^+}{I}} \overline{y})(x,\alpha)$$

$$= \frac{1}{\Gamma(n-\beta)} (\frac{d}{d_x})^n \int_a^x \frac{\overline{y}(t,\alpha)}{(x-t)\beta-n+1} dt \quad (4-1-3)$$

Lemma 4-2

Reimann – liouvill fuzzy fractional derivatue of order $\beta \in C(R(\beta) \geq 0$ de fried y

$$({}^{C\beta}_{b^-}D y)(x,\alpha) = (\frac{-d}{d_x})^n ({}^{n-\beta}_{b^-}I\, y)(x,\alpha)$$

$$= \frac{1}{\Gamma(n-\beta)} (-\frac{d}{d_x})^n \int_x^b y(t,\alpha) \Big/ (t-x)^{\beta-n+1} dt \quad (x<b) \qquad (4-2-1)$$

$$({}^{C\beta}_{b^-}D \bar{y})(x,\alpha) = (-\frac{d}{d_x})^n ({}^{n-\beta}_{b^-}I\, \bar{y})(x,\alpha)$$

$$= \frac{1}{\Gamma(n-\beta)} (-\frac{d}{d_x})^n \int_x^b \frac{\bar{y}(t,\alpha)}{(t-x)^{\beta-n+1}} dt \quad (x<b) \qquad (4-2-2)$$

Respect tirely the fuzzy fractional derivatire Caputo

$({}^{C\beta}_{ca^+}D y)(x,\alpha)$ *and* $({}^{C\beta}_{b^-}D y)(x,\alpha)$ *of order*

$\beta \in C$, $(R(\beta) \geq 0)$ *on* $[a,b]$ *are*

Defined via the above ca puto fra ctional derivative by

$$({}^{C\beta}_{a^+}D y)(x,\alpha) = {}^{\beta}_{b^-}D[y(t,\alpha) - \sum_{k=0}^{n-1} \frac{y(a,\alpha)}{k!}(t-a)^k]$$

$$({}^{C\beta}_{b^-}D y)(x,\alpha) = {}^{\beta}_{b^-}D[y(t,\alpha) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b,\alpha)}{k!}(b-t)^k] \qquad (4-2-4)$$

Theorem 4-1

Let $0 < R(\beta) < 1$ *and* $y(x) \in C[a,b]$

$$(D^{\beta}_{a+}y)(x,r) = \left(I^{1-\beta}_{b_-}Dy\right)(x,r) \quad ,(D^{\beta}_{b-}y)(x,r) = \left(I^{1-\beta}_{b_-}Dy\right)(x,r)$$

Theorenm 4-2

Let $R(\beta) \geq 0$ *and* Let n be

Given $n = [R(\beta)] + 1$ for

$r\beta \notin N_0$ $n = \beta$ *for* $\beta \in N_0$

Also let $y(x) \in c^n[a,b]$ them caputo fuzzy fractional derivative $({}^{C\beta}_{a^+}D y)(x,\alpha)$ and $({}^{C\beta}_{b^-}D y)(x,\alpha)$ are continuous on [a, b]

$({}^{C\beta}_{a^+}D y)(x,\alpha)$ and $({}^{C\beta}_{b^-}D y)(x,\alpha) \in C[a,b]$

Them

$$({}^{C\beta}_{a^+}D y)(a,\alpha) = ({}^{C\beta}_{b^-}D y)(b,\alpha) = 0 \qquad (4-2-5)$$

In particular , then have respectively the forms the orem (4-1) and (4-2) for $0 < R(\beta) < 1$

Pro of:

Let $\beta \notin N_0$ Formulas

$$({}^{C\beta}_{b^-}D \bar{y})(x,\alpha) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{y^{(n)}(t,\alpha)}{(n-t)^{\beta-n+1}} dt = ({}^{n\beta}_{a^+}I\, D^n y)(x,\alpha) \qquad (4-2-7)$$

$$({}^{C\beta}_{b^-}D y)(x,\alpha) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \frac{y^{(n)}(t,\alpha)dt}{(t-x)^{\beta-n+1}} = (-1)^{(n)} ({}^{n\beta}_{b^-}I\, D^n y)(x,\alpha)$$

are Proved as in theorem (4-1) the continuity of the function $({}^{C\beta}_{a^+}D y)(x,\alpha)$ and follows from the repre sentations (4-1),(4-2) according to Lemma 4-1 , with

$f[t, y(t)]^{\alpha} = y^{(n)}(t,\alpha)$

$f[t, y(t,\alpha)] \in C[a,b]$ *and* $\beta = 0$ *the*

Relations (4-2-5) follow from the following in tequalities

$$\left|({}^{n-\beta}_{a^+}I\, D^n y)(x,\alpha)\right| \leq \frac{\left\|y^{(n)}(t,\alpha)\right\|c}{\left|\Gamma(n-\beta)\right|[n-R(\beta)+1]}(x-\alpha)^{n-R(\beta)} \qquad (4-2-9)$$

and

$$\left|({}^{n-\beta}_{b^-}I\, D^n y)(x,\alpha)\right| \leq \frac{\left\|y^{(n)}(t,\alpha)\right\|c}{\left|\Gamma(n-\beta)\right|[n-R(\beta)+1]}(b-x)^{n-R(\beta)} \qquad (4-2-10)$$

Whieh are yalid any $x \in [a,b]$ and Proved directly a sing (3-1-5) and (3-1-4)

when $\beta \in N_0$ the first relation in a sing (3-1-5) and

$$({}^{C\beta}_{a^+}D y)(x,\alpha) = y^{(n)}(x,\alpha)$$

$$({}^{C\beta}_{b^-}D y)(x,\alpha) = (-1)^n y^{(n)}(x,\alpha)$$

5- FFIES Under Caputo GH-differentiability

We consider the following fuzzy Caputo fractional differential equation:

$$(D^{C}_{a+})_{\beta}U(t) = F(t, \lambda u(t)) \qquad (D^{C}_{a+})_{\beta-1}u(t_0) = u_0^{\beta-1} \in E \qquad (5\text{-}1)$$

Where $F: (a,b) * E \to E$ is continuous fuzzy -valued function and $t_0 \in [a,b]$. The following Lemma transform the fuzzy fractional differential equations to the corresponding fuzzy Volterra integral equations.

Lemma 5-1

Let $r \in [0,1]$ *and* $t_0 \in R$, *the fuzzy fractional differential equation (5-1) is equivalent to one of the following integral equations*

$$U(t) = u(t_0) + \frac{\lambda}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(s,u(s))ds}{(t-s)^{1-\beta}}, t \in [a,b] \qquad (5\text{-}2)$$

if U is differentiable,and

$$U(t) = u(t_0) !_{GH} \frac{-\lambda}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(s,u(s))ds}{(t-s)^{1-\beta}}, t \in [a,b] \qquad (5\text{-}3)$$

and U is $[-ii-\beta]^c -$ differentiable,provided that the H-difference exists.

Proof. Let us consider $f \in C^F[a,b]$, then we have following

$(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f(t;r)=[(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f^{-}(t;r),(I_{a+}^{\beta}D_{a+}^{c}f_{-}(t;r)],r\in[0,1]$ (5-4)

$(I_{a+}^{\beta}D_{a+}^{c})f^{-}(t;r)=f^{-}(t;r)-f^{-}(t_0;r),(I_{a+}^{\beta}D_{a+}^{c})f_{-}(t;r)=f_{-}(t;r)-f_{-}(t_0;r)$ (5-5)

For case $[-ii-\beta]^{c}$-differentiability. For case $[-i-\beta]^{c}$-differentiability, We have

$(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f(t;r)=[(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f_{-}(t;r),(I_{a+}^{\beta}D_{a+}^{c})f^{-}(t;r)],$ (5-6)

Finally we recall that for case $[-i-\beta]^{c}$-differentiability,

$(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f(t;r)=[f_{-}(t;r)-f_{-}(t_0;r),f^{-}(t;r)-f^{-}(t_0;r)],$ (5-7)

and also case $[-ii-\beta]^{c}-$ differentiability,

$(I_{a+}^{\beta}D_{a+}^{c})_{\beta}f(t;r)=[f^{-}(t;r)-f^{-}(t_0;r),f_{-}(t;r)-f_{-}(t_0;r)],$ (5-8)

which completes the proof [8].

Theorem 5-1 *[4]*

We consider the following fuzzy Caputo fractional differential equation

$$(D_{a+}^{C})_{\beta}U(t)-\lambda*c(r)*u(t)=f(t)$$ (5-9)

let $f:[a,b]*(a,b)*E\rightarrow E$ be bounded continuous functions.Let the sequens $u_n:[a,b]\rightarrow E$ is given by

$$\lim_{t\rightarrow o+}(t^{1-\beta}D_{0+}^{C})U(t)=u_0^{1-\beta}\in E$$ (5-10)

$r\in[0,1]$,$\beta\in(0,1]$, $\lambda\in R$ has a unique solution given by (49)

$U(t)=\frac{1}{\Gamma(\beta)}(u_{0+}^{\beta-1}t^{\beta-1}E_{\beta,\beta}(\lambda t^{\beta}))+\frac{\lambda}{\Gamma(\beta)}\int_{o+}^{t}\frac{f(s)ds}{(t-s)^{1-\beta}}E_{\beta,\beta}(\lambda(t-s)^{\beta}),$ (5-11)

For case $[-ii-\beta]^{c}$-differentiability and

$U(t)=\frac{1}{\Gamma(\beta)}(u_{0+}^{\beta-1}t^{\beta-1}E_{\beta,\beta}(\lambda t^{\beta}))!\frac{-\lambda}{\Gamma(\beta)}\int_{o+}^{t}\frac{f(s)ds}{(t-s)^{1-\beta}}E_{\beta,\beta}(\lambda(t-s)^{\beta}),$ (5-12)

Theorem 5-2 *[4]*

Let $f:[a,b]\rightarrow E$ *be a fuzzy-valued function on* $[a,b]$

- f is $[-ii-GH]^{C}$-differentiable at $C\in[a,b]$ iff f is $[-ii-GH]^{CF}$-differentiable at C.
- f is $[-i-GH]^{C}$-differentiable at $C\in[a,b]$ iff f is $[-i-GH]^{CF}$-differentiable at C.

Lemma 5-2

Let $f:[a,b]\rightarrow E$ *be a fuzzy-valued function such that* $F_{GH}^{\prime L}\in C^{F}[a,b]\cap\ L^{F}[a,b]$,

$$(I_{a+}^{\beta}D_{a+}^{\beta})_{GH}^{L}f(t)=f(t)!_{GH}f(t_0)=I_{a+}^{1-\beta}f_{GH}^{\prime L}(t),$$ (5-13)

Proof:By using Definition (2-3) and(2-4) we have $(I_{a+}^{\beta}f')_{GH}(t)=(I_{a+}^{\beta}D_{a+}^{\beta})_{GH}f(t)=\int_a^b f'_{GH}(S)ds,$ Such that

$$\int_a^b f'_{GH}(S)ds=I_{a+}^{\beta}I_{a+}^{1-\beta}f'_{GH}(t)$$ (5-14)

We consider f is $[-i-GH]^{Cf}$- differentiable. according Theorem (5-2) f is $[-i-GH]^{C}$- differentiable.Then we have

$$\int_a^b f'_{GH}(S)ds=[I_{a+}^{\beta}I_{a+}^{1-\beta}f'_{GH}(t)]=(I_{a+}^{\beta})_{GH}f(t)$$ (5-15)

$(I_{a+}^{\beta}D_{a+}^{\beta})_{GH}f(t)=[\int_a^b(f')_{\beta}^{-}(s)ds,\int_a^b(f')_{\beta}^{+}(s)ds]=f_{\beta}(t)!_{GH}f_{\beta}(t_0),$ (5-16)

according Theorem (4)f is $[-ii-GH]^{Cf}$-differentiable.Then we have

$(I_{a+}^{\beta}D_{a+}^{\beta})_{GH}f(t)=[\int_a^b(f')_{\beta}^{+}(s)ds,\int_a^b(f')_{\beta}^{-}(s)ds]=f_{\beta}(t)!_{GH}f_{\beta}(t_0),$ (5-17)

For all $t\in[a,b]$, $r\in[0,1]$, $\beta\in(0,1]$, which proves the lem.

Theorem5-3

Let $f:[a,b]*E*E\rightarrow E$ *be a fuzzy-valued function such that* $F_{GH}^{\prime L}\in C^{F}[a,b]\cap\ LF[a,b]$, *Let the sequens* $u_n:[a,b]\rightarrow E$ *is given by*

$$u_0(t)=u_0,U_{n+1}(t)=u_0(t)!_{GH}\frac{-\lambda}{\Gamma(\beta)}\int_{t_0}^{t}\frac{f(s,u_n(s))}{(t-s)^{1-\beta}}ds$$ (5-18)

is defined for any $n\in N$.Then the sequens u_n is convex sentence to unique solution of problem (59) which is $[-ii-GH]^{cf}$-differentiable on $[a,b]$,provided that $\lambda<1$.

Proof. Now we show that sequence u_n, (5-18) is a Cauchy sequence $in C^{F}[a,b]$. To do this end,We have

$$d(u_1,u_0)=d(u_0!\frac{-\lambda}{\Gamma(\beta)}\int_{t_0}^{t}(\frac{f(s,u_0(s))}{d}s(t-s)^{1-\beta},u_0)$$
$$\leq\frac{\lambda}{\Gamma(\beta)}\int(t-s)^{\beta-1}d(f(s,u_0(s)),0^{\sim})=\lambda t_0^{\beta}M$$ (5-19)

Where $M=\sup d(f(s,u(s)),o^{\sim})$. Since f is Lipschitz continuous, so by Definition (2-4), we can find that Suppose that $d(u_n(s),u_{n-1}(s))\leq\mu_{n-1}$, then using assumption, we have

$$d(u_{n+1}(s),u_n(s)) = \frac{\lambda}{\Gamma(\beta)} d\left(\int_{t_0}^{t} (t-s)^{\beta-1} f(s,u_n(s))ds, (t-s)^{\beta-1} f(s,u_{n-1}(s))\right)$$
$$\leq \frac{\lambda}{\Gamma(\beta)} \int_{t_0}^{t} d((t-s)^{\beta-1} f(s,u_n(s)), (t-s)^{\beta-1} f(s,u_{n-1}(s)))ds$$
(5-20)

$$d(u_{n+1}(s),u_n(s)) \leq \frac{\lambda}{\Gamma(\beta)} \int_{t_0}^{t} ((t-s)^{\beta-1} g(s,d(u_n(s),u_{n-1}(s)))ds$$
(5-21)

$$d(u_{n+1}(s),u_n(s)) = \mu_n(s).$$
(5-22)

Moreover $|(D_{a+}^{c})_{\beta} u_{n+1}(t)|_{=}|g(s,u_n(s)| \leq M_1$; and therefore,we can conclude by Ascoli-Arzela theorem and the monotonicity of the sequence u_n that $\lim_{n\to\infty} \mu_n(t) = \mu(t)$ uniformly on $[t_0, t_0 + r]$ and

$$\mu(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \frac{g(s,u(s))ds}{(t-s)^{1-\beta}}$$
(5-23)

Thus ,by the inductive method ,We know

$$d(u_{n+1}(s),u_n(s)) \leq \mu_n(s),$$
(5-24)

$\forall t \in [t_0, t_0 + r]$, $n = 0,1,2,3,\dots$ so ,we have

$$d((D_{a+}^{c})_{\beta} u_{n+1}(t),(D_{a+}^{c})_{\beta} u_n(t)) = d(f(s,u_n(s),f(s,u_{n-1}(s)))$$
$$\leq g(s,d(u_n(s),u_{n-1}(s))).$$
(5-25)

$$d((D_{a+}^{c})_{\beta} u_{n+1}(t),(D_{a+}^{c})_{\beta} u_n(t)) \leq g(s,d(u_n(s),u_{n-1}(s)))$$
(5-26)

Examples

[1]- consider the following FF-IDE

$$(D_{a+}^{C})_{\beta} U(t) - \lambda U(t) = F(t, c(r).u(t)),$$
(5-27)

$$\lim_{t\to o+} t^{1-\beta} D_{0+}^{C} u(t) = u_0^{1-\beta} \in E \qquad (5\text{-}28)$$

Where $\lambda > 0$ and we use $[-i - GH]^{C}$-differentiability. So, Eq (5-9) is equivalent to the following fuzzy Caputo fractional integral-differential equations by applying Theorem (5-2). finally, by applying of Mittag-Leffler function $E_{\beta,\beta}(t)$, we get the following

$$U(t) = \frac{1}{\Gamma(\beta)} (u_{0+}^{\beta-1} t^{\beta-1} E_{\beta,\beta}(\lambda t^{\beta}))$$
$$+ \frac{1}{\Gamma(\beta)} \int_{0+}^{t} \frac{F(s,u(s))ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^{\beta})$$
$$+ \frac{\lambda}{\Gamma(\beta)} \int \frac{u(s)ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^{\beta})$$
(5-29)

For case $[-ii - \beta]^{c}$-differentiability,this positive solution and assumption $\lambda > 0$ such that

$$U(t) = \frac{1}{\Gamma(\beta)} (u_{0+}^{\beta-1} t^{\beta-1} E_{\beta,\beta}(\lambda t^{\beta}))$$
$$! \frac{-1}{\Gamma(\beta)} \int_{0+}^{t} \frac{F(s,u(s))ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^{\beta})$$
$$! \frac{-\lambda}{\Gamma(\beta)} \int_{0+}^{t} \frac{u(s)ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^{\beta})$$
(5-30)

In order to solve mentioned fuzzy Volerra integral equation,We adopt successive Approximation method. We set

$$U_0(t) = \frac{1}{\Gamma(\beta)} (u_{0+}^{\beta-1} c) \qquad (5\text{-}31)$$

$$U_{n+1}(t) = U_0(t) !_{GH} \frac{-1}{\Gamma(\beta)} \int_{0+}^{t} \frac{F(s,u_n(s))ds}{(t-s)^{1-\beta}} ! \frac{-\lambda}{\Gamma(\beta)} \int_{0+}^{t} \frac{u_n(s))ds}{(t-s)^{1-\beta}}.$$
(5-32)

[2] -Consider the fuzzy Fredholm integral equation with

$$\underline{f}(t,r) = rt + \frac{۳}{۲۶} - \frac{۳}{۲۶} r - \frac{۱}{۱۳} r - \frac{۱}{۱۳} t^{۲} r$$

$$\bar{f}(t,r) = ۲t - rt + \frac{۳}{۲۶} r + \frac{۱}{۱۳} t^{۲} r - \frac{۳}{۲۶} - \frac{۳}{۱۳} t^{۲}$$

and kernel

$$K(s,t) = \frac{s^{۲} + t^{۲} - ۲}{۱۳}$$

$$\circ \leq s,t \leq ۲$$

and a =0, b =2. The exact solution in this case is given by

$$\underline{u}(t,r) = rt$$
$$\bar{u}(t,r) = (۲ - r)t$$

Some first terms of Adomian decomposition series are

$$\underline{u_\circ} = rt + \frac{۳}{۲۶} - \frac{۳}{۲۶} r - \frac{۱}{۱۳} r - \frac{۱}{۱۳} t^{۲} r$$

$$\underline{u_۱} = \frac{۲۹}{۳۳۸} r - \frac{۴۴۹}{۵۰۷۰} + \frac{۱۱}{۱۶۹} t^{۲} r + \frac{۲۹}{۵۷۰} t^{۲}$$

and

$$\underline{u_۲} = \frac{-۲۳۵}{۱۹۷۷۳} + \frac{۵۱}{۲۱۹۷} r + \frac{۱۲۹۸}{۹۸۸۵۶} t^{۲} + \frac{۲۲}{۲۱۹۷} t^{۲} r$$

$$\underline{u_\circ{}'} = g = rt + \frac{۵۰۲}{۹۸۸۵۶} - \frac{۱۴}{۲۱۹۷} r - \frac{۶۵۲}{۹۸۸۵۶} t^{۲} - \frac{۴}{۲۹۱۷} t^{۲} r$$

$$\underline{u_۱{}'} = \frac{۱۴۰۴۰۶۷}{۱۲۸۵۴۵۰} - \frac{۳۰۹۹۱}{۲۸۵۶۱۰} r - \frac{۲۷۲۳۷}{۳۸۵۵۷۲} t^{۲} + \frac{۶۶۳۱}{۸۵۶۸۳} t^{۲} r$$

$$\underline{u_۲{}'} = \frac{-۱۲۷۹۸۳}{۷۵۱۶۵۲} + \frac{۲۹۲۰۴۲}{۱۶۷۰۸۱۵} r + \frac{۱۷۰۱۷۳}{۷۵۱۶۸} t^{۲} - \frac{۱۳۶۷۹}{۱۶۷۸۱۵۸} t^{۲} r$$

[3] -Consider the fuzzy Fredholm integral equation with

$$\underline{f}(r,t) = e^{rt}$$

$$\bar{f}(r,t) = ۴^{t} - r^{۲} + ۱$$

and kernel

$$K(s,t) = \frac{s^{۲} + t^{۲} - ۲}{۱۳} \qquad \circ \leq S,t \leq ۲$$

and a =0, b =2. The exact solution in this case is given by

Some first terms of Adomian decomposition series are

$$K(s,t)=\begin{cases}\dfrac{s^{۲}+t^{۲}-۲}{۱۳} & ۰\le s\le t \quad ۰\le t\le ۲\\ ۰ & \text{otherwise}\end{cases}$$

Some first terms of Adomi es are

$$\begin{cases}\underline{u}_{۰}=-۲e^{rt}\\ \underline{u}_{۱}=\dfrac{۵۲۰}{۱۷۵۰}e^{-rt}+\dfrac{۳}{۴۰۵۰}e^{۲rt}\\ \underline{u}_{۲}=\dfrac{۱۷۵۰}{۶۵۹۷۰}e^{-rt}+\dfrac{۵۶۷}{۱۶۸۰}e^{-۲rt}+\dfrac{۳۵۰}{۷۰۷۰}e^{-۳rt}\end{cases}$$

$$\begin{cases}\overline{u}_{۰}=\dfrac{۵}{۱۷۸۰}e^{rt}\\ \overline{u}_{۱}=\dfrac{-۵۲۰}{۱۷۵۰}e^{rt}-\dfrac{۳}{۴۰۵۰}e^{۲rt}\end{cases}$$

6- Conclusions

Adomian's method is relatively straightforward to apply at least with the assistance of a powerful Computer Algebra Package and, in simple cases, produces a series that can converge rapidly to known solution [7].

As shown in the previous section, for particular parameter values in our Hammerstein integral equation ,Adomian's method appears to show rapid convergence to the unique solution obtained using the contraction mapping principle. The accuracy of Adomian's method has been further con®rmed by comparison with a numerical solution of the original boundary value problem obtained using a shooting method. This result con®rms the view expressed by Some [5] who compared various numerical methods for solving fuzzy integral equations and concluded that Adomian's method was fast and e • cient. we will obtain positive solution of FFIES with fuzzy Caputo H-Differentiability and fuzzy caputo Hukuhara differentiability which is used to investigate convergence of this set of equations.

References

[1] D.Dubios and H.Prade , Towards fuzzy differential Calculus , Part 1. Integration of fuzzy mapping , Fuzzy sets and systems 8(1982),1 -17

[2] B .Bede, L. Stefanini, *Generalizations differentiability of to fuzzy-valued functions*, fuzzy Set and Systems, 2013, 230: 119-141.

[3.] R.Goetschel and W.Voxman , Flementary calculus , fuzzy sets and systems 18(1986),31-43

[4] T. Allahviranloo, S. Abbasbandy, S. Salahshour, *Fuzzy fractional differential equations with Nagumo and Krasnoselskii-Krein condition,* In: EUSFLAT-LFA 2011, july 2011, Aix-les-Bains, France.

[5] B. Some, Some recent numerical methods for solving nonlinear Hammerstein integral equations, Mathematical Computer Modelling 18 (1993) 55±62.

[6] R.P.Agarwal , M.Benchohra, D.ORegan and A.Ouahab, ,fuzzy solutions for multi –point boundary ralue problems , mem , Differential Equotions math . Phys .35 (2005),1-14.

[7] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Dordrecht, 1994. [8] M.A.Darwish and A.A.El-Bary , Existence of fractional integral equation with hysteresis , Appl.Math .Comput . 176(2006) , 684-687

[9]. M.A.Darwish , on quadratic integral equation of fractional orders , J.Math. Anal.APPl .311 (2005) , 112-119.

[10].R.Hilifer , Applications of Fractional calculus in physics , world scientific, Singapore, 2000

[11] K.B. Oldham and J.Spanier , The Frac tional Calculus :Theory and applications of Differen tiation and Integration to Arbitary order, Academie press , New York and Lndon , 1974

[12]. I .Podlubny , Fractional Differential Equations , Academic Press , New York , and London , 1990.

[13] . S.G. Samko , A.A . Kilbas and O.I.Marichev, Fractional Integrals and Derivatives:

[14] .A.A.Kilbas , Hari M.Srivastava , and Juan J. Trujillo , Theory and Applications of Fractional Differential Equations North –Holland Mathematics Studies, 204 Elsevier Science B.V. Amsterdam, 2006

[15]. I. Podlubny; Fractional Differential Equation , Academic Press , San Diego , 1999

[16] Y-K. Chang, J.J.Nieto; Some new existence Yesults for fractional differential inclusions with boundary conditions, Math. Comput. Modelling 49(2009)605-609.

[17] O.Kaleva; A note on fuzzy differential equations, Nonlinear Analysis (64)2006, (895-200)

[18] V.Lakshmikant ham, R.N.Mohapatra; Theory of fuzzy differential equations and inclusions, Taylor 8 Francis, London, 2003