Spline Collocation for Volterra - Fredholm
Integral Equations

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ABSTRACT

The purpose of this paper is to develop a numerical method based on quintic B-spline to solve the linear and nonlinear Volterra-Fredholm integral equations. The solution is collocated by quintic B-spline and then the integrand is approximated by the Newton-Cotes formula. The arising system of linear or nonlinear algebraic equations can solve the linear combination coefficients appearing in the representation of the solution in spline basic functions. The error analysis of proposed numerical method is studied theoretically.

AMS Subject Classifications 41A15, 65R20.

1. Introduction

Currently

Consider the linear and nonlinear Volterra-Fredholm integral equations of the form

\[ y(t) = g(t) + \int_{a}^{t} k_1(t,s)y(s)ds + \int_{a}^{t} k_2(t,s)y(s)ds, \quad t \in [a,b]. \]  

(1)

The given kernels \( k_1, k_2 \) are continuous on \([a, b]\) and satisfy a uniform Lipschitz, and \( f(t) \) is the known function and \( y \) is unknown function. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. The existence and the uniqueness are discussed and given in Refs. [2] and [8]. The nonlinear Volterra-Fredholm integral equation (1) arises from various physical and biological models. The essential features of these models are of wide applicable [1], [3], [4] and [10]. Several numerical methods for approximating the solution of nonlinear Volterra- Fredholm integral equations are known. The numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method was introduced in [5]. Minggen et al. [7], used the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations in the reproducing kernel space. The exact solution is given by the form of series. Its approximate solution is obtained by truncating the series and a new numerical approximate method. Ordokhani [12,15], applied the rationalized Haar functions to approximate of the nonlinear Volterra- Fredholm-Hammerstein integral equations. Hendi in [14] used collocation and Galerkin methods. Mirzaee et al. [13], used hybrid of block-pulse functions and Taylor method. Also, in [11], Yalcinbas developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations. Using a global approximation to the solution of Fredholm and Volterra integral equation of the second kind is constructed by means of the spline quadrature in [6,13, 17-19]. In this paper we will develop a collocation method based on quintic B-spline to approximate the unknown function in equation (1) then, the Newton-Cotes formula is used to approximate the linear and nonlinear Fredholm - Volterra integral equations of second kind.

2. Quintic B-spline

We introduce the quintic B-spline space and basis functions to construct an interpolation \( s \) to be used in the formulation of the quintic B-spline collocation method. Let \( \pi_{i} = \{a = t_0 < t_1 < \cdots < t_n = b\} \) be a uniform partition of the interval \([a, b]\) with step size \( h = \frac{b-a}{n} \). The quintic spline space is denoted by \( S_{5}(a) = \{s \in C^4[a, b] | s | [t_{l-1}, t_{l}] \in P_5, \quad l = 0,1, \ldots, N\} \), where \( P_5 \) is the class of quintic polynomials. The construction of the quintic B-spline interpolate \( s \) to the analytical solution \( y \) for (1) can be performed with the help of the ten additional knots such that

\[ t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1} \text{ and } t_{5} < t_{4} < t_{13} < t_{12} < t_{15} < t_{14} < t_{15} . \]

Following [10] we can define a quintic B-spline \( s(t) \) of the form

\[ s(t) = \sum_{k \in S} c_k B^k(t), \]

(2)

where \( c_k \) are the unknown coefficients. The unknown coefficients \( c_k \) are determined by requiring the quintic B-spline \( s(t) \) to satisfy the integral equation (1) at the knots of the partition \( \pi_{i} \). The arising system of linear or nonlinear algebraic equations can solve the linear combination coefficients appearing in the representation of the solution in spline basic functions. The error analysis of proposed numerical method is studied theoretically.
3. The Collocation Method

3.1 Nonlinear Volterra-Fredholm integro-differential equation

In the given nonlinear Volterra-Fredholm integral Eq. (1), we can approximate the unknown function by quintic B-spline (2), then we obtain:

\[ s(\tau) = g(\tau) + \int_{a}^{\tau} k_1(\tau, x) s(x) \, dx + \int_{a}^{\tau} k_0(\tau, x) s(x) \, dx, \quad \tau \in [a, b]. \]  

(4)

We now collocate Eq. (4) at collocation points \( \tau_j = a + jh, \ h = \frac{b-a}{N}, \ j = 0, 1, \ldots, N \), and we obtain

\[ s(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i), \quad j = 1, \ldots, N. \]  

(5)

To approximate the integral Eq. (5), we can use the Newton-Cotes formula, when \( n \) is even then the Simpson rule can be used and when \( n \) is multiple of 3, we have to use the three-eighth rule, then we get the following nonlinear system

\[ s(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i), \quad j = 1, \ldots, N. \]  

(6)

where \( x_i = a + ih, i = 0, \ldots, N \). We need more equations to obtain the unique solution for Eq. (6). Hence by associating Eq. (6) with (3), we have the following nonlinear system

\[(N + 5) \times (N + 5): \]

\[ s(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i), \quad j = 1, \ldots, N. \]

\[ s(t_i) = g(t_i) + \sum_{j=0}^{N} w_{ij} k_1(t_i, x_j) s(x_j), \quad j = 1, 2, 3, 4. \]  

(7)

where \( W_{ji} \) represents the weights for a quadrature rule of Newton-Cotes type. By solving the above nonlinear system, we can determine the coefficients \( C_i, i = -1, \ldots, N + 1 \). By setting \( C_i \) in (2), we obtain the approximate solution for Eq. (1).

4. Error analysis: convergence of the approximate solution

In this section, we consider the error analysis of the Volterra-Fredholm integral equation of the second kind. To obtain the error estimation of our approximation, first we recall the following definition in [10].

**Definition**: The most immediate error analysis for spline approximates \( \mathcal{S} \) to a given function \( f \) defined on an interval \([a, b]\) follows from the second integral relations.

If \( f \in C^n[a, b] \), then

\[ \left\| D^j (f - \mathcal{S}) \right\| \leq \gamma h^{n-j}, f = 0, \ldots, n. \]  

Where \( \left\| f \right\|_m = \max_{x \in [a, b]} \sup_{x \in [a, b]} |f(x)| \),

and \( D^i \) the j-th derivative.

**Theorem**: The approximate method

\[ s(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i), \quad j = 1, \ldots, N. \]  

(8)

for solution of the nonlinear Volterra-Fredholm integral Eq. (4) is converge and the error bounded is

\[ |e_j| \leq \varepsilon_{n+1} \sum_{i=0}^{N} \| e_i \| + hW_{L+1} \sum_{i=0}^{N} \| e_i \|. \]  

**Proof**: We know that at

\[ t_j = a + jh, h = \frac{b-a}{N}, \ j = 1, \ldots, N \]  

and the corresponding approximation method for nonlinear Volterra-Fredholm integral Eq. (4) is

\[ s(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i), \quad j = 1, \ldots, N. \]  

By discretizing (1) and approximating the integrand by the Newton-Cotes formula, we obtain

\[ y(t_j) = g(t_j) + \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i) + \#(h t_j), j = 1, \ldots, N. \]  

(10)

where

\[ \#(h, t_j) = \sum_{i=0}^{N} w_{ji} k_1(t_j, x_i) s(x_i) + \sum_{i=0}^{N} w_{ji} k_0(t_j, x_i) s(x_i). \]

By subtracting (10) from (9) and using interpolatory conditions of quintic B-spline, we get

\[ s(t_j) - y(t_j) = h \sum_{i=0}^{N} w_{ji} [k_1(t_j, x_i) s(x_i)] - k_1(t_j, x_i) s(x_i) + h \sum_{i=0}^{N} w_{ji} [k_2(t_j, x_i) s(x_i)] - k_2(t_j, x_i) s(x_i)]. \]

We suppose that \( W = \max_{i,j} \| w_{ji} \| \) and

\[ s(t_j) = s_{ij}(y(t_j) = y_j, j = 1, \ldots, N \text{, and kernels } k_1, k_2 \text{ satisfy a Lipschitz condition in its third argument of the form}\]

\[ \| k_1(t, x, y) - k_2(t, x, y) \| \leq L |s(y) - y|, \| k_2(t, x, y) - k_2(t, x, z) \| \leq L^* |s(y) - y|, \]

where \( L, L^* \) are independent of \( t, x, s \) and \( y \). We get
Then we have
\[
|e_j| \leq hWL \sum_{i=0}^{n} \left| s(x_i) - y(x_i) \right| + hWL \sum_{i=0}^{n} \left| s(x_i) - y(x_i) \right|
\]
When \( h \to 0 \) then the above first and second terms are zero. We get for a fixed \( j \),
\[
|e_j| \to 0 \text{ as } h \to 0.
\]

5. Conclusion

In the present work, a technique has been developed for solving the linear and nonlinear Volterra-Fredholm integral equations by using the Newton-Cotes formula and collocating by quintic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions.

References