University College of Takestan

Volume 2,Issue 1

Available online at **http://UCTjournals.com**

UCT Journal of Research in Science ,Engineering and Technology

UCT . J. Resea. Scien. Engineer.Techno.(UJRSET)

2(1): *04-07(2014)*

ISSN:2382-977X

Spline Collocation for Volterra - Fredholm Integral Equations

***Nehzat Ebrahimi*[*1], *Jalil Rashidinia*[2]**

[1,2] *Department of Mathematics,Islamic Azad University,Central Tehran Branch,Iran*

*Corresponding author's E-mail: Ebrahimi_nehzat@yahoo.com

ABSTRACT

The purpose of this paper is to develop a numerical method based on quintic B-spline to solve the linear and nonlinear Volterra-Fredholm integral equations. The solution is collocated by quintic B-spline and then the integrand is approximated by the Newton-Cotes formula. The arising system of linear or nonlinear algebraic equations can solve the linear combination coefficients appearing in the representation of the solution in spline basic functions. The error analysis of proposed numerical method is studied theoretically.

Original Article:

Received 30 Dec. 2013
Accepted 17 Feb. 2014
Published 30 Mar. 2014

Keywords:
Volterra-Fredholm, integral equations, quintic B-spline , Newton-Cotes formula, Error analysis

AMS Subject Classifications 41A15, 65R20.

1.Introduction

Currently

Consider the linear and nonlinear Volterra-Fredholm integral equations of the form

$$y(t) = g(t) + \int_a^t k_1(t,x,y(x))dx + \int_a^b k_2(t,x,y(x))dx\,, \qquad t \in [a,b]. \qquad (1)$$

The given kernels k_1, k_2 are continuous on [a, b] and satisfie a uniform Lipschitz, and $g(t)$ is the known function and *y* is unknown function. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. The existence and the uniqueness are discussed and given in Refs. [2] and [8]. The nonlinear Volterra-Fredholm integral equation (1) arises from various physical and biological models. The essential features of these models are of wide applicable [1], [3], [4] and [10]. Several numerical methods for approximating the solution of nonlinear Volterra- Fredholm integral equations are known. The numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method was introduced in [5]. Minggen et al. [7], used the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations in the reproducing kernel space. The exact solution is given by the form of series. Its approximate solution is obtained by truncating the series and a new numerical approximate method. Ordokhani [12,15], applied the rationalized Haar functions
to approximate of the nonlinear Volterra- Fredholm-Hammerstein integral equations .Hendi in [14] used collocation and Galerkin methods . Mirzaee et al. [13], used hybrid of block-pulse functions and Taylor series method .Also, in [11], Yalcinbas developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations. Using a global approximation to the solution of Fredholm and Volterra integral equation of the second kind is constructed by means of the spline quadrature in [6,13, 17-19].
In this paper we will develop a collocation method based on quintic B-spline to approximate the unknown function in equation (1) then, the Newton-Cotes formula is used to approximate the linear and nonlinear Fredholm - Volterra integral equations of second kind.

2. Quintic B-spline

We introduce the quintic B-spline space and basis functions to construct an interpolation s to be used in the formulation of the quintic B-spline collocation method. Let $\pi: \{a = t_0 < t_1 < \cdots < t_N = b\}$, be a uniform partition of the interval $[a,b]$ with step size $h = \frac{b-a}{N}$. The quintic spline space is denoted by

$$S_5(\pi) = \{s \in C^4[a,b];\, s|[t_i,t_{i+1}] \in P_5 \quad , \quad i = 0,1,..,N\},$$

where P_5 is the class of quintic polynomials. The construction of the quintic B-spline interpolate s to the analytical solution y for (1) can be performed with the help of the ten additional knots such that

$$t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1} \text{ and } t_{N+1} < t_{N+2} < t_{N+3} < t_{N+4} < t_{N+5}\,.$$

Following $[10]$ we can define a quintic B-spline s(t) of the form

$$s(t) = \sum_{i=-2}^{N+2} c_i B_i^5(t), \qquad (2)$$

where

$$B_i(t) = \frac{1}{120h^5}\begin{cases}(t-t_{i-3})^5 & ,t\in[t_{i-3},t_{i-2}]\\ (t-t_{i-3})^5-6(t-t_{i-2})^5 & ,t\in[t_{i-2},t_{i-1}]\\ (t-t_{i-3})^5-6(t-t_{i-2})^5+15(t-t_{i-1})^5 & ,t\in[t_{i-1},t_i]\\ (t-t_{i-3})^5-6(t-t_{i-2})^5+15(t-t_{i-1})^5-20(t-t_i)^5 & ,t\in[t_i,t_{i+1}]\\ (t-t_{i-3})^5-6(t-t_{i-2})^5+15(t-t_{i-1})^5-20(t-t_i)^5+15(t-t_{i+1})^5 & ,t\in[t_{i+1},t_{i+2}]\\ (t-t_{i-3})^5-6(t-t_{i-2})^5+15(t-t_{i-1})^5-20(t-t_i)^5+15(t-t_{i+1})^5-6(t-t_{i+2})^5 & ,t\in[t_{i+2},t_{i+3}]\\ 0 & , otherwise,\end{cases}$$

satisfying the following interpolator conditions:

$$s(t_i)=y(t_i)\,, \qquad 0\le i\le N,$$

and the end conditions

$(i) s^j(t_0)=y^j(t_0)\,,\quad s^j(t_N)=y^j(t_N),\quad j=1,2,$

or

$(ii) D^j s(t_0)=D^j s(t_N)\,,\quad j=1,2,3,4,$ (3)

or

$(iii) s^j(t_0)=0\ \ ,s^j(t_N)=0\,,\qquad j=3,4\,.$

3. The Collocation Method

3.1 Nonlinear Volterra-Fredholm integro-differential equation

In the given nonlinear Volterra- Fredholm integral Eq. (1), we can approximate the unknown function by quintic B-spline (2), then we obtain:

$$s(t)=g(t)+\int_a^t k_1(t,x,s(x))dx+\int_a^b k_2(t,x,s(x))dx\,,\quad t\in[a,b]. \tag{4}$$

We now collocate Eq. (4) at collocation points $t_j=a+jh, h=\frac{b-a}{N}\ ,\ j=0,1,\dots,N$, and we obtain

$$s(t_j)=g(t_j)+\int_a^{t_j} k_1\left(t_j,x,s(x)\right)dx+\int_a^b k_2\left(t_j,x,s(x)\right)dx\,,\quad j=1,\dots,N\ . \tag{5}$$

To approximate the integral Eq. (5), we can use the Newton- Cotes formula , when n is even then the Simpson rule can be used and when n is multiple of 3 ,we have to use the three-eighth rule, then we get the following nonlinear system:

$$s(t_j)=g(t_j)+h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\ ,j=1,\dots,N. \tag{6}$$

where $x_i=a+ih, i=0,\dots,N$,we need more equations to obtain the unique solution for Eq. (6). Hence by associating Eq. (6) with (3) , we have the following nonlinear system $(N+5)\times(N+5)$:

$$\begin{cases} s(t_j)=g(t_j)+h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,s(x_i)\right) & ,j=1,\dots,N,\\ s(t_0)=g(t_0)+h\sum_{i=0}^{N} w_{0,i}k_2(t_0,x_i,s(x_i)), & \\ D^j s(t_0)=D^j s(t_N)\,,\quad j=1,2,3,4, & \end{cases} \tag{7}$$

where $w_{j,i}$ represents the weights for a quadrature rule of Newton-Cotes type. By solving the above nonlinear system , we can determine the coefficients $c_i, i=-1,\dots,N+1$, by setting c_i in (2), we obtain the approximate solution for Eq. (1).

4. Error analysis: convergence of the approximate solution

In this section, we consider the error analysis of the Volterra-Fredholm integral equation of the second kind .To obtain the error estimation of our approximation, first we recall the following definition in [10].

Definition: The most immediate error analysis for spline approximates s to a given function f defined on an interval $[a,b]$ follows from the second integral relations.

If $f\in C_6[a,b]$, then $\|D^j(f-S)\|\le\gamma h^{6-j}, j=0,\dots,6.$ Where $\|f\|_\infty=\max_{0\le i\le N} sup_{t_{i-1}\le t\le t_i}|f(t)|,$

and D^j the j-th derivative.

Theorem : The approximate method

$$s(t_j)=g(t_j)+h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\ ,j=1,\dots,N, \tag{8}$$

for solution of the nonlinear Volterra- Fredholm integral Eq .(4) is converge and the error bounded is

$$|e_j|\le hWL\sum_{i=0}^{j}|e_i|+hWL^*\sum_{i=0}^{N}|e_i|.$$

Proof : We know that at $t_j=a+jh, h=\frac{t-a}{N}\ ,\ j=1,\dots,N$,the corresponding approximation method for nonlinear Volterra- Fredholm integal Eq. (4) is

$$s(t_j)=g(t_j)+h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\quad ,j=1,\dots,N. \tag{9}$$

By discretizing (1) and approximating the integrand by the Newton- Cotes formula, we obtain

$$y(t_j)=g(t_j)+h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,y(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,y(x_i)\right)+E(h,t_j),\ j=1,\dots,N. \tag{10}$$

Where

$$E(h,t_j)=\left(\int_a^{t_j} k_1\left(t_j,x,y(x)\right)dx+\int_a^b k_2\left(t_j,x,y(x)\right)dx\ \right)-\left(h\sum_{i=0}^{j} w_{j,i}k_1\left(t_j,x_i,y(x_i)\right)+h\sum_{i=0}^{N} w_{j,i}k_2\left(t_j,x_i,y(x_i)\right)\right).$$

By subtracting (10) from (9) and using interpolatory conditions of quintic B-spline, we get

$$s(t_j)-y(t_j)=h\sum_{i=0}^{j} w_{j,i}[k_1\left(t_j,x_i,s(x_i)\right)-k_1\left(t_j,x_i,y(x_i)\right)]+h\sum_{i=0}^{N} w_{j,i}[k_2\left(t_j,x_i,s(x_i)\right)-k_2\left(t_j,x_i,y(x_i)\right)].$$

We suppose that $W=\max_{i,j}|w_{j,i}|$ and $s(t_j)=s_j, y(t_j)=y_j,\ j=1,\dots,N$. and kernels k_1,k_2 satisfy a Lipschitz condition in its third argument of the form

$$|k_1(t,x,s)-k_1(t,x,y)|\le L|s-y|, |k_2(t,x,s)-k_2(t,x,y)|\le L^*|s-y|,$$

where L, L^* are independent of t, x, s and y. We get

$$|s_j - y_j| \leq hWL \sum_{i=0}^{j} (s(x_i) - y(x_i))] + hWL^* \sum_{i=0}^{N} (s(x_i) - y(x_i))].$$

Then we have

$$|e_j| \leq hWL \sum_{i=0}^{j} |e_i| + hWL^* \sum_{i=0}^{N} |e_i| \quad .$$

Where $e_j = s_j - y_j \ \ , j = 1, \dots, N.$

When $h \to 0$ then the above first and second terms are zero. We get for a fixed j,

$$|e_j| \to 0 \ \ as\, h \to 0 \, .$$

5.Conclusion

In the present work, a technique has been developed for solving the linear and nonlinear Volterra-Fredholm integral equations by using the Newton-Cotes formula and collocating by quintic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions.

References

[1] Bakodah, H.O., Darwish, M.A., 2012, On discrete Adomian decomposition method with Chebyshev abscissa for nonlinear integral equations of Hammerstein Type, Advances in Pure Mathematics, 2 310-313.

[2] Brunner, H., 1990, On the numerical solution of nonlinear Volterra-Fredholm integral equation by collocation methods, SIAM J. Numer. Anal., 27 987-1000.

[3] Chuong, N.M., Tuan, N.V., 1996, Spline collocation methods for a system of nonlinear Fredholm- Volterra integral equations. Acta Math. Vietnamica 21 , 155-169.

[4] Darwish, M.A., 1999, Fredholm-Volterra integral equation with singular kernel, Korean J. Comput. Appl. Math., 6 163-174.

[5] Darwish, M.A., 1999, Note on stability theorems for nonlinear mixed integral equations, Korean J. Comput. Appl. Math., 6 633-637.

[6] Ghasemi, M., Tavassoli Kajani, M., Babolian, E., 2007, Numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method, Appl. Math. Comput., 188 446-449.

[7] Hendi, F.A., Albugami, A.M., 2010, Numerical solution for Fredholm-Volterra integral equation of the second kind by using collocation and Galerkin methods, Journal of King Saud University (Science) ,22, 37-40.

[8] Mahmoodi, Z., Rashidinia, J., Babolian, E., 2012, B-Spline collocation method for linear and nonlinear Fredholm and Volterra integro-differential equations. Applicable Analysis, 1-16.

[9] Minggen, C., Hong, D., 2006, Representation of exact solution for the nonlinear Volterra-Fredholm integral equations, Appl. Math. Comput., 182 1795- 1802.

[10] Mirzaee, F., Hoseini, A.A., 2013, Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series ,Alexandria Engineering Journal, 52, 551-555.

[11] Netravali, A.N., Figueiredo, R.J.P., 2003, Spline approximation to the solution of the linear Fredholm integral equation of the second kind .SIAM. J. Numer. Anal.11, 1974,538-549.

[12] Ordokhani, Y., 2006, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions, Appl. Math. Comput., 180 436-443.

[13] Ordokhani Y., M. Razzaghi , 2008, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions, Applied Mathematics Letters ,21, 4-9.

[14] Pachpatte, B. G., 2008, On a nonlinear Volterra-Fredholm integral equation, Sarajevo J. Math., 16, 61-71.

[15] Prenter, P.M., 1975, Spline and Variational Methods. Wiley & Sons, New-York.

[16] Rashed M.T., An expansion method to treat integral equations. Appl. Math. Comput.135, 65-72.

[17] Rashidinia, J., Babolian, E., Mahmoodi, Z., 2011, Spline Collocation for Fredholm Integral Equations. Mathematical Sciences,5 (2), 147-158.

[18] Rashidinia, J., Babolian, Mahmoodi, E. Z., 2011, Spline Collocation for nonlinear Fredholm Integral Equations. International Journal of Mathematical Modelling and Computations, 1 (1) 69-75.

[19] Wazwaz, A. M., 2002, A reliable treatment for mixed Volterra-Fredholm integral equations, Appl. Math. Comput., 127, 405-414.

[20] Yalcinbas S., Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations, Appl. Math. Comput., 127 (2002), 195-206.