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Spline Collocation for Volterra - Fredholm Integro-Differential Equations

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ABSTRACT

A collocation procedure is developed for the linear and nonlinear Volterra - Fredholm integro-differential equations, using the globally defined B-spline and auxiliary basis functions. The solution is collocated by cubic B-spline and the integrand is approximated by the Newton-Cotes formula. The error analysis of proposed numerical method is studied theoretically.

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1.Introduction

Consider the nonlinear Volterra-Fredholm integro-differential equation of the form

$$\sum_{r=0}^{m} p_r(t)\, y^{(r)}(t) = g(t) + \int_a^t k_1(t,x,y(x))\,dx + \int_a^b k_2(t,x,y(x))\,dx\,, m = 1,2, t \in [a,b], \qquad (1)$$

with the boundary conditions,

$$\sum_{r=0}^{m-1} [\alpha_{i,r} y^{(r)}(a) + \beta_{i,r} y^{(r)}(b)] = \gamma_i\,, \qquad 0 \le i \le m-1, \qquad (2)$$

where $\alpha_{i,r}, \beta_{i,r}$ and γ_i are given real constants. The given kernels $k_1, k_2$are continuous on [a, b] and satisfie a uniform Lipschitz, and $g(t)$ and $p_r(t)$ are the known functions and *y* is unknown function. The boundary value problems in terms of integro-differential equations have many practical applications. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. Of course, these equations can also appear when transforming a differential equation into an integral equation [1, 4, 5].Due to this, some authors have proposed numerical methods to approximate the solutions of nonlinear Fredholm-Volterra integro-differential equations. To mention a few, in [8] the authors have discussed the Taylor polynomial method for solving integro-differential equations (1).The triangular functions method has been applied to solve the same equations in [3].Furthermore, the operational matrix with block-pulse functions method is carried out in [2] for the aforementioned integro-differential equations. The Hybrid Legendre polynomials and Block- Pulse functions approach for solving integro-differential equations (1) are proposed in [7]. Yalcinbas in [15] developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations and in [14] considered the high-order linear Volterra-Fredholm integro-differential equations. Using a global approximation to the solution of Fredholm and Volterra integral equation of the second kind is constructed by means of the spline quadrature in [6, 10, 9, 11, 12, 13].

In this paper, we use cubic B-spline collocation for approximation unknown function and use of the Newton-Cotes rules for approximating integrand.

2. cubic B − spline

We introduce the cubic B-spline space and basis functions to construct an interpolation s to be used in the formulation of the cubic B-spline collocation method. Let $\pi: \{a = t_0 < t_1 < \cdots < t_N = b\}$, be a uniform partition of the interval $[a, b]$ with step size $h = \frac{b-a}{N}$. The cubic spline space is denoted by

$$S_3(\pi) = \{s \in C^2[a,b]; s|[t_i, t_{i+1}] \in P_3 \quad , \quad i = 0,1,\ldots,N\},$$

where P_3 is the class of cubic polynomials. The construction of the cubic B-spline interpolate s to the analytical solution y for (1) can be performed with the help of the four additional knots such that

$$t_{-2} < t_{-1} < t_0 \;\; and \;\; t_N < t_{N+1} < t_{N+2}\,.$$

We can define a cubic B-spline s(t) of the form

$$s(t) = \sum_{i=-1}^{N+1} c_i B_i^3(t),$$

where

$$B_i(t)=\frac{1}{6h^3}\begin{cases}(t-t_{i-2})^3 & , \quad if\ t\in[t_{i-2},t_{i-1}]\\ h^3+3h^2(t-t_{i-1})+3h(t-t_{i-1})^2-3(t-t_{i-1})^3, & if\ t\in[t_{i-1},t_i]\\ h^3+3h^2(t_{i+1}-t)+3h(t_{i+1}-t)^2-3(t_{i+1}-t)^3, & if\ t\in[t_i,t_{i+1}]\\ (t_{i+2}-t)^3 & , \quad if\ t\in[t_{i+1},t_{i+2}]\\ 0 & , \quad otherwise,\end{cases}$$

satisfying the following interpolator conditions:

$$s(t_i)=y(t_i)\,,\qquad 0\le i\le N,$$

and the end conditions

$$(i)\,s'(t_0)=y'(t_0)\,,\qquad s'(t_N)=y'(t_N),$$

or

$$(ii)\,D^js(t_0)=D^js(t_N)\,,\quad j=1,2, \qquad (4)$$

or

$$(iii)\,s''(t_0)=0\ \ ,s''(t_N)=0\,.$$

3. The Collocation Method

3.1 Nonlinear Volterra-Fredholm integro-differential equation

In the given nonlinear Volterra-Fredholm integro-differential Eq. (1), we can approximate the unknown function by cubic B-spline (3), then we obtain:

$$\sum_{r=0}^{m}p_r(t)s^{(r)}(t)=g(t)+\int_a^t k_1(t,x,s(x))dx+\int_a^b k_2(t,x,s(x))dx\,,m=1,2,t\in[a,b],\ (5)$$

with the boundary conditions,

$$\sum_{r=0}^{m-1}\left[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)\right]=\gamma_i\,,\qquad 0\le i\le m-1.$$

We now collocate Eq. (5) at collocation points $t_j=a+jh,h=\frac{b-a}{N}\,,\ j=0,1,\dots,N\,,$ and we obtain

$$\sum_{r=0}^{m}p_r(t_j)s^{(r)}(t_j)=g(t_j)+\int_a^{t_j}k_1\left(t_j,x,s(x)\right)dx+\int_a^b k_2\left(t_j,x,s(x)\right)dx\,,m=1,2,j=1,\dots,N\,.\ (6)$$

To approximate the integro-differential Eq. (6), we can use the Newton- Cotes formula , when n is even then the Simpson rule can be used and when n is multiple of 3 ,we have to use the three-eighth rule, then we get the following nonlinear system:

$$\sum_{r=0}^{m}p_r(t_j)s^{(r)}(t_j)=g(t_j)+h\sum_{i=0}^{j}w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N}w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\,,j=1,\dots,N,m=1,2,\ (7)$$

with the boundary conditions,

$$\sum_{r=0}^{m-1}\left[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)\right]=\gamma_i\,,\qquad 0\le i\le m-1,$$

where $x_i=a+ih,i=0,\dots,N$,we need more equations to obtain the unique solution for Eq. (7). Hence by associating Eq. (7) with (4) , we have the following nonlinear system $(N+3)\times(N+3)$:

$$\begin{cases}\sum_{r=0}^{m}p_r(t_j)s^{(r)}(t_j)=g(t_j)+h\sum_{i=0}^{j}w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N}w_{j,i}k_2\left(t_j,x_i,s(x_i)\right) & ,j=1,\dots,N,m=1,2,\\ \sum_{r=0}^{m}p_r(t_0)s^{(r)}(t_0)=g(t_0)+h\sum_{i=0}^{N}w_{j,i}k_2(t_0,x_i,s(x_i)) & \\ \sum_{r=0}^{m-1}\left[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)\right]=\gamma_i\,, & 0\le i\le m-1,\end{cases}\qquad(8)$$

where $w_{j,i}$ represents the weights for a quadrature rule of Newton-Cotes type. By solving the above nonlinear system , we can determine the coefficients $c_i,i=-1,\dots,N+1$, by setting c_i in (3), we obtain the approximate solution for Eq. (1).

4. Error analysis: convergence of the approximate solution

In this section, we consider the error analysis of the Volterra-Fredholm integro-differential equation of the second kind .To obtain the error estimation of our approximation, first we recall the following definition in [10].

Definition : The most immediate error analysis for spline approximates s to a given function f defined on an interval $[a,b]$ follows from the second integral relations.

If $f\in C_4[a,b]$, then $\|D^j(f-S)\|\le\gamma h^{4-j},j=0,1,2,3,4.$ Where $\|f\|_\infty=\max_{0\le i\le N}sup_{t_{i-1}\le t\le t_i}|f(t)|$,

and D^j the j-th derivative.

Theorem : The approximate method

$$\sum_{r=0}^{m}p_r(t_j)s^{(r)}(t_j)=g(t_j)+h\sum_{i=0}^{j}w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N}w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\quad ,j=1,\dots,N,m=1,2.\ \ (9)$$

for solution of the nonlinear Volterra- Fredholm integro-differential Eq .(5) is converge and the error bounded is

$$\left|e_j^{(m)}\right|\le\frac{hWL}{|p_{mj}|}\sum_{i=0}^{j}|e_i|+\frac{hW\,L^*}{|p_{mj}|}\sum_{i=0}^{N}|e_i|\ +\frac{1}{|p_{mj}|}\sum_{r=0}^{m-1}|p_{rj}|\left|e_j^{(r)}\right|$$

Proof : We know that at $t_j=a+jh,h=\frac{t-a}{N}\,,\ j=1,\dots,N\,,$the corresponding approximation method for nonlinear Volterra- Fredholm integro-differential Eq. (5) is

$$\sum_{r=0}^{m}p_r(t_j)s^{(r)}(t_j)=g(t_j)+h\sum_{i=0}^{j}w_{j,i}k_1\left(t_j,x_i,s(x_i)\right)+h\sum_{i=0}^{N}w_{j,i}k_2\left(t_j,x_i,s(x_i)\right)\quad ,j=1,\dots,N,m=1,2.\ \ (10)$$

By discretizing (1) and approximating the integrand by the Newton- Cotes formula, we obtain

$$\sum_{r=0}^{m}p_r(t_j)y^{(r)}(t_j)=g(t_j)+h\sum_{i=0}^{j}w_{j,i}k_1\left(t_j,x_i,y(x_i)\right)+h\sum_{i=0}^{N}w_{j,i}k_2\left(t_j,x_i,y(x_i)\right)+E(h,t_j),\ j=1,\dots,N,m=1,2.\qquad(11)$$

Where

$$E(h,t_j) = (\int_a^{t_j} k_1\left(t_j, x, y(x)\right) dx + \int_a^b k_2\left(t_j, x, y(x)\right) dx\ ) - \left(h \sum_{i=0}^{j} w_{j,i} k_1\left(t_j, x_i, y(x_i)\right) + h \sum_{i=0}^{N} w_{j,i} k_2\left(t_j, x_i, y(x_i)\right)\right).$$

By subtracting (11) from (10) and using interpolatory conditions of cubic B-spline, we get

$$\sum_{r=0}^{m} p_r(t_j)\,[s^{(r)}(t_j) - y^{(r)}(t_j)]$$
$$= h \sum_{i=0}^{j} w_{j,i}[k_1\left(t_j, x_i, s(x_i)\right) - k_1\left(t_j, x_i, y(x_i)\right)] + h \sum_{i=0}^{N} w_{j,i}[k_2\left(t_j, x_i, s(x_i)\right) - k_2\left(t_j, x_i, y(x_i)\right)].$$

We suppose that $W = \max_{i,j} |w_{j,i}|$ and
$s^{(m)}(t_j) = s_j^{(m)}, y^{(m)}(t_j) = y_j^{(m)},\ j = 1, \ldots, N\ , m = 1,2.$

and kernels k_1, k_2 satisfy a Lipschitz condition in its third argument of the form

$$|k_1(t,x,s) - k_1(t,x,y)| \le L|s-y|,\ |k_2(t,x,s) - k_2(t,x,y)| \le L^*|s-y|,$$

where L, L^* are independent of t, x, s and y. We get

$$|p_{mj}|\,\left|s_j^{(m)} - y_j^{(m)}\right| \le hWL \sum_{i=0}^{j} (s(x_i) - y(x_i))] + hWL^* \sum_{i=0}^{N} (s(x_i) - y(x_i))] + \sum_{r=0}^{m-1} |p_{rj}|\left|s_j^{(r)} - y_j^{(r)}\right| \quad ,j = 1, \ldots, N.$$

Since that $|p_{mj}| \neq 0,$ then we have

$$|e_j^{(m)}| \le \frac{hWL}{|p_{mj}|} \sum_{i=0}^{j} |e_i| + \frac{hWL^*}{|p_{mj}|} \sum_{i=0}^{N} |e_i| + \frac{1}{|p_{mj}|} \sum_{r=0}^{m-1} |p_{rj}|\,|e_j^{(r)}| \ .$$

Where $e_j^{(m)} = s_j^{(m)} - y_j^{(m)} \ ,j = 1, \ldots, N, r = 0, \ldots, m.$

When $h \to 0$ then the above first and second term are zero and the third term in the above tends to zero because this term is due to interpolating of $y(t)$ by cubic B-spline. We get for a fixed j,

$$|e_j^{(m)}| \to 0 \ as\ h \to 0\ , m = 0,1,2.$$

5.Conclusions

In the present work, a technique has been developed for solving linear and nonlinear Volterra- Fredholm integro-differential equations by using the Newton-Cotes formula and collocating by cubic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination

coefficients appearing in the representation of the solution in spline basic functions.

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